ANALYTICAL GEOMETRY FOR HIGH SCHOOLS

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ANALYTICAL GEOMETRY

FOR

HIGH SCHOOLS

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The Parabola Section
1926

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1927

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FORMULÆ

The following important results from algebra and trigonometry are frequently used in analytical geometry:—

1. The roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

These roots are

real,	if $b^2 >$ or $= 4ac$;
imaginary,	if $b^2 < 4ac$;
equal to each other,	if $b^2 = 4ac$;
equal in magnitude but opposite in sign,	if $b = 0$;
rational,	if $b^2 - 4ac$ is a perfect square.
One root $= 0$,	if $c = 0$;
both roots $= 0$,	if $b = c = 0$.

The sum of the roots $= \dfrac{-b}{a}$.

The product of the roots $= \dfrac{c}{a}$.

2. The fraction $\dfrac{a}{b} = \infty$, if $b = 0$ and a is not $= 0$.

3. The equation $ax + by + c = 0$ is the same as $px + qy + r = 0$,

$$\text{if } \frac{a}{p} = \frac{b}{q} = \frac{c}{r}.$$

4. $ax^2 + bx + c$ is a perfect square, if $b^2 = 4ac$.

5. If $a_1x + b_1y + c_1z = 0$

and $a_2x + b_2y + c_2z = 0$,

$$\text{then } \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}.$$

6. For all values of a,

$$\sin^2 a + \cos^2 a = 1.$$

?

7. If $\tan a = k$, $a = \tan -{}^1k$.

8. $\sin(\mathrm{A} \pm \mathrm{B}) = \sin \mathrm{A} \cos \mathrm{B} \pm \cos \mathrm{A} \sin \mathrm{B}.$
$\cos(\mathrm{A} \pm \mathrm{B}) = \cos \mathrm{A} \cos \mathrm{B} \mp \sin \mathrm{A} \sin \mathrm{B}.$

9. $\sin \mathrm{A} + \sin \mathrm{B} = 2 \sin \frac{\mathrm{A}+\mathrm{B}}{2} . \cos \frac{\mathrm{A}-\mathrm{B}}{2}$, etc.

10. $\tan(\mathrm{A} \pm \mathrm{B}) = \frac{\tan \mathrm{A} \pm \tan \mathrm{B}}{1 \mp \tan \mathrm{A} \tan \mathrm{B}}.$

$\tan 45^\circ = 1$

$\therefore 45 = \tan^{-1} 1$

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ELEMENTARY ANALYTICAL GEOMETRY

CHAPTER I

CARTESIAN COORDINATES

1. **Analytical, or algebraic, geometry** was invented by Descartes in 1637, and this invention marks the beginning of the history of the modern period of mathematics. It differs from pure geometry in that it lays down a general method, in which, by a few simple rules, any property can be at once proved or disproved, while in the latter each problem requires a special method of its own.

2. **The Origin.** In plane analytical geometry the positions of all points in the plane are determined by their distances and directions as measured from a fixed point.

If the points are all in a st. line, the fixed point is most conveniently taken in that line.

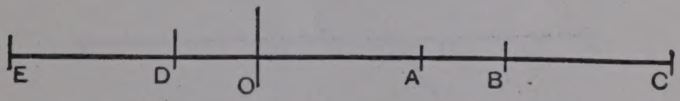


FIG. 1.

Thus, if the distance and direction of each of the points A, B, C, D, E from the point O are given, the positions of these points are known.

The point O is called the **origin,** or **pole.**

3. Use of plus and minus. In algebra the signs plus and minus are used to indicate opposite qualities of the numbers to which they are prefixed; and in analytical geometry, as in trigonometry, these signs are used to show difference of direction. In a horizontal st. line distances measured from the origin to the right are taken to be positive, while those to the left are negative; and in a vertical st. line distances measured upward are positive, while those measured downward are negative. Thus, in Fig. 1, if OA = 2 cm., OB = 3 cm., OC = 5 cm., OD = 1 cm., and OE = 3 cm., the positions of these points are respectively represented by 2, 3, 5, − 1 and − 3, the understood unit being one centimetre.

RECTANGULAR COORDINATES

4. Coordinates. When points are not in the same st. line, their positions are determined by their distances

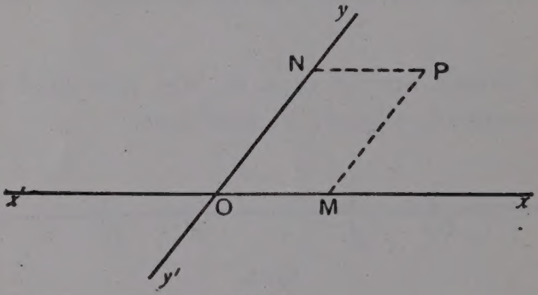


FIG. 2.

from two st. lines $x'\mathrm{O}x$ and $y'\mathrm{O}y$ drawn through the origin, the distances being measured in directions ∥ to the given st. lines.

These lines are called the **axes of coordinates,** or shortly, the **axes.**

$x'Ox$ is called the axis of x, and $y'Oy$ is called the axis of y.

From a point P draw $PM \parallel Oy$ and $PN \parallel Ox$, terminated in the axes.

PM is called the **ordinate** of P, and PN (= OM), is called the **abscissa** of P. These two distances, the abscissa and ordinate, are called the **coordinates** of the point.

Sometimes, from the name of the inventor, they are spoken of as **cartesian coordinates.**

5. **Rectangular coordinates.** When the axes are at rt. ∠s to each other, the distances of a point from the axes are called its **rectangular coordinates.**

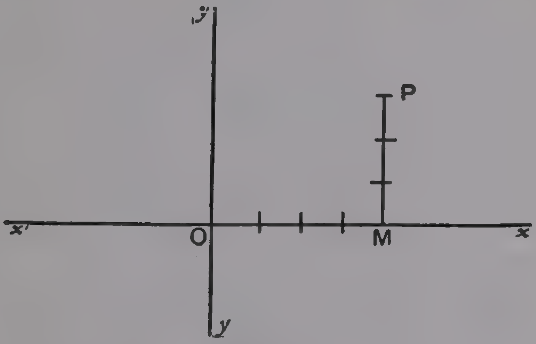


Fig. 3.

To locate the point of which the abscissa is 4 and the ordinate 3 when the coordinates are rectangular, measure the distance OM = 4 units along Ox and at M erect the ⊥ PM = 3 units. P is the required point.

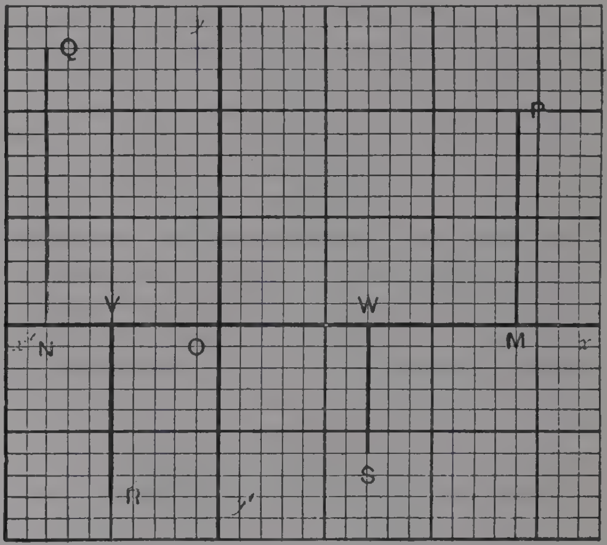


FIG. 4. (Unit = ½ inch.)

In Fig. 4, the abscissa of P = OM = 2·8, the ordinate of P = PM = 2. The position of this point is then indicated by the notation (2·8, 2). For Q, the abscissa = ON = − 1·6, the ordinate = QN = 2·6 and the position of the point is indicated by (−1·6, 2·6).

Similarly the position of R is (−1, −1·6), and that of S is (1·4, −1·2).

xOy, yOx', x'Oy' and y'Ox are respectively called the first, second, third and fourth quadrants; and we see from the diagram, that:—

for a point in the first quadrant both coordinates are positive;

for a point in the second quadrant the abscissa is negative and the ordinate is positive;

for a point in the third quadrant both are negative; and

for a point in the fourth the abscissa is positive and the ordinate is negative.

Thus the signs of the coordinates show at once in which quadrant the point is located.

6.—Exercises

1. Write down the coordinates of the points **A**, **B**, **C**, **D**, **E**, **F**, **G**, **H** and **O** in Fig. 5.

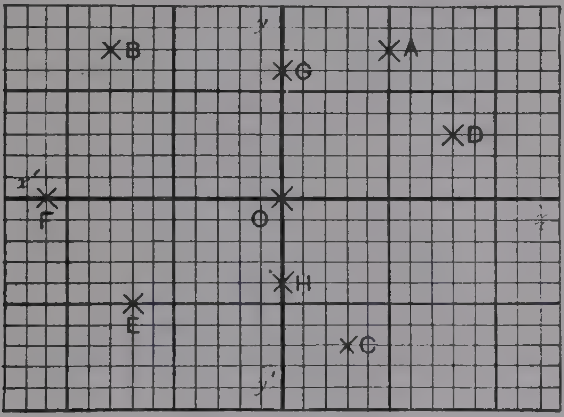


Fig. 5. (Unit = $\frac{1}{10}$ inch.)

2. Draw a diagram on squared paper, and mark on it the following points:—**A** (4, 3), **B** (4·6, 0), **C** (– 2, – 3), **D** (– 4, 2), **E** (0, 2·8). Indicate the unit of measurement on the diagram.

3. Draw a diagram on squared paper and mark the following points:—(4, 3), (3, 4), (– 3, – 4), (– 4, 3), (0, 5), (0, – 5), (– 5, 0). Describe a circle with centre **O** and

radius 5. Should the circle pass through the seven points? Why?

4. The side of an equilateral △ = $2a$. One vertex is at the origin, one side is on the axis of x and the △ is in the first quadrant. What are the coordinates of the three vertices?

5. One corner of a square is taken as origin and the axes coincide with two sides. The length of a side is b. What are the coordinates of the corners, the square being in the first quadrant?

THE DISTANCE BETWEEN TWO POINTS

7. In general, the abscissa of a point is represented by x, the ordinate by y.

8. **To find the distance between a point P (x_1, y_1) and the origin.**

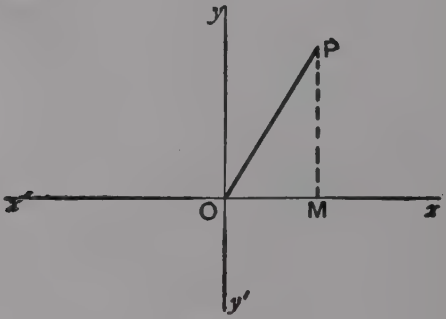


FIG. 6.

From P draw PM ⊥ Ox.

$$\because \text{PMO is a rt.-}\angle\text{d } \triangle,$$

$$\therefore PO^2 = OM^2 + PM^2$$

$$= x_1^2 + y_1^2.$$

$$\therefore PO = \sqrt{x_1^2 + y_1^2}.$$

9. To find the distance between P (x_1, y_1) and Q (x_2, y_2).

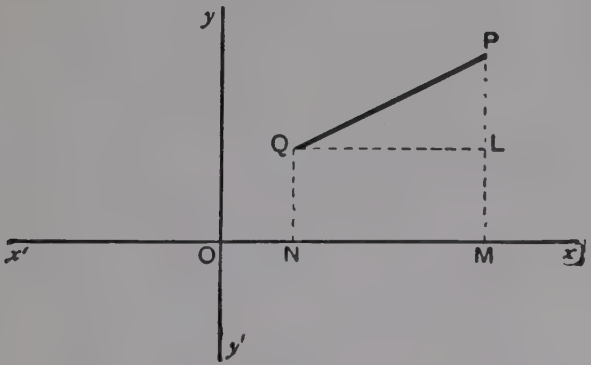


FIG. 7.

Draw PM and QN ⊥ Ox; QL ⊥ PM.

QL = NM = OM − ON = $x_1 - x_2$.

PL = PM − LM = PM − QN = $y_1 - y_2$.

∵ PLQ is a rt.-∠d △,

∴ $PQ^2 = QL^2 + PL^2$.

$$= (x_1 - x_2)^2 + (y_1 - y_2)^2.$$

$$\therefore PQ = \sqrt{(\mathbf{x}_1 - \mathbf{x}_2)^2 + (\mathbf{y}_1 - \mathbf{y}_2)^2}.$$

10. If the point Q in § 9 coincides with the origin O, $x_2 = 0$ and $y_2 = 0$. Substituting these values, in the expression for PQ in that article we obtain

$$PO = \sqrt{x_1^2 + y_1^2}.$$

This shows that the result in § 8 is a particular case of that in § 9.

11. The result in § 9 holds good, *in the same form*, for any two points whether the coordinates are positive or negative.

For example—it is required to find the distance between P $(-3, 2)$ and Q $(5, -2)$.

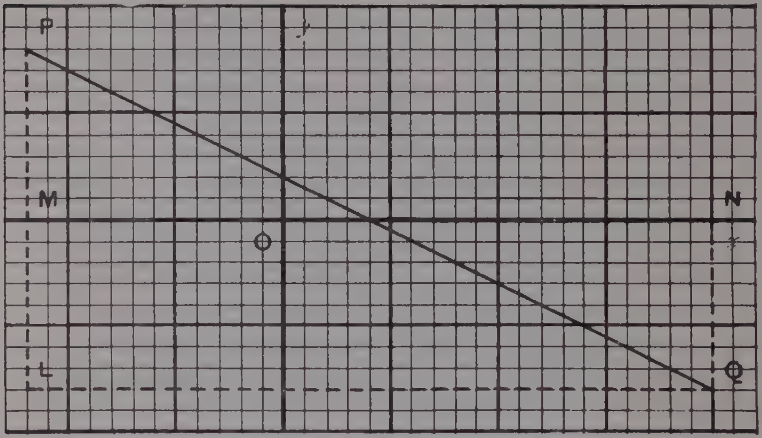


Fig. 8. (Unit = $\frac{4}{16}$ inch.)

Draw PM, QN $\perp$ Ox; QL $\perp$ PM.

The length of ML = length of NQ = 2.

$$\therefore \quad \text{PL} = \text{PM} + \text{ML} = 2 + 2 = 4.$$

The length of QL = NM = $5 + 3 = 8$.

$$\text{PQ}^2 = \text{QL}^2 + \text{PL}^2$$
$$= 64 + 16 = 80.$$
$$\therefore \quad \text{PQ} = 4\sqrt{5}.$$

If in the expression for PQ found in § 9, we substitute -3 for x_1, 2 for y_1, 5 for x_2 and -2 for y_2, we obtain

$$\text{PQ} = \sqrt{(-3-5)^2 + (2+2)^2}$$
$$= 4\sqrt{5},$$

the same result.

12. The particular cases in § § 10 and 11 illustrate what is known as the **continuity** of the formulæ in analytical geometry. Here continuity means, that *general* results which are obtained when the coordinates in the diagram used are all positive hold true in the same form for all points.

13. To find the coordinates of the middle point of the distance between two given points P (x_1, y_1) and Q (x_2, y_2).

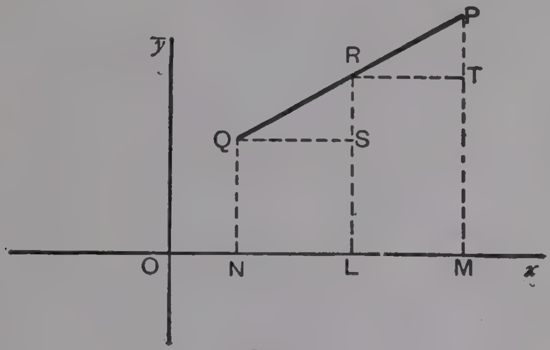


FIG. 9.

Let R (x, y) be the middle point of PQ.

Draw PM, QN, RL ⊥ Ox; QS ⊥ RL; RT ⊥ PM.

From the equality of △s PRT, RQS,

$$QS = RT \text{ and } RS = PT,$$

$$\therefore NL = LM,$$

$$\therefore \quad x - x_2 = x_1 - x.$$

$$\therefore \quad x = \frac{x_1 + x_2}{2}.$$

$$\because RS = PT,$$

$$\therefore \quad y - y_2 = y_1 - y.$$

$$\therefore \quad y = \frac{y_1 + y_2}{2}.$$

Thus the coordinates of R are

$$\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}.$$

14. To find the coordinates of the point dividing the distance between P $(x_1\ y_1)$ and Q $(x_2\ y_2)$ in the ratio of *m* to *n*.

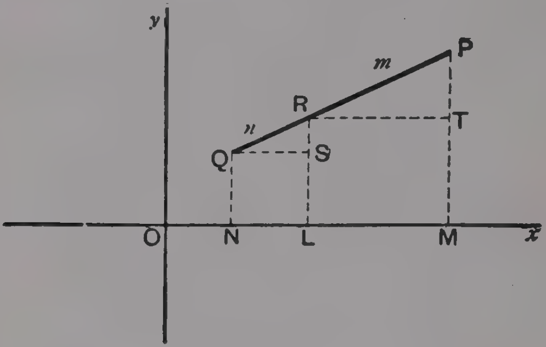


FIG. 10.

Let R (x, y) be the point dividing PQ such that $\frac{\text{PR}}{\text{RQ}} = \frac{m}{n}$.

Draw PM, QN, RL ⊥ Ox; QS ⊥ RL; RT ⊥ PM.

From the similar △s PRT, RQS

$$\frac{\text{RT}}{\text{QS}} = \frac{\text{PT}}{\text{RS}} = \frac{\text{PR}}{\text{RQ}} = \frac{m}{n}.$$

$$\because \quad \frac{\text{RT}}{\text{QS}} = \frac{m}{n},$$

$$\therefore \quad \frac{x_1 - x}{x - x_2} = \frac{m}{n}.$$

$$\therefore \quad mx - mx_2 = nx_1 - nx.$$

$$\therefore \qquad x = \frac{nx_1 + mx_2}{m + n}.$$

$$\because \qquad \frac{\mathsf{PT}}{\mathsf{RS}} = \frac{m}{n},$$

$$\therefore \quad \frac{y_1 - y}{y - y_2} = \frac{m}{n}.$$

$$\therefore \qquad my - my_2 = ny_1 - ny.$$

$$\therefore \qquad y = \frac{ny_1 + my_2}{m + n}.$$

Thus the coordinates of R are

$$\frac{nx_1 + mx_2}{m + n}, \ \frac{ny_1 + my_2}{m + n}.$$

15. If the point R be taken in PQ produced such that PR : RQ $= m : n$, and the coordinates of P, Q be (x_1, y_1), (x_2, y_2) it may be shown by a proof similar to that in the previous article that the coordinates of R are

$$\frac{mx_2 - nx_1}{m - n}, \ \frac{my_2 - ny_1}{m - n}.$$

These results and also those of § § 13 and 14 are the same for oblique and rectangular axes.

16.—**Exercises**

1. Find the distance between the points (6, 5) and (1, –7) and test your result by measurement on squared paper.

2. Find the distance between the points (2, –3) and (–1, 1) and test your result by measurement on squared paper.

3. Find the coordinates of the middle points of the st. lines joining the pairs of points in exercises 1 and 2 respectively and test the results by measurements on the diagrams.

4. Find, to two decimal places, the distance between $(-3, 7)$ and $(4, -4)$.

5. The vertices of a △ are $(-2, 4)$, $(-8, -4)$ and $(7, 4)$. Find the lengths of its sides.

6. The vertices of a △ are $(-1, 5)$, $(-4, -2)$, $(5, -3)$. Find (*a*) the lengths of the sides; (*b*) the lengths of the medians.

7. The vertices of a quadrilateral are $(4, 3)$, $(-5, 2)$, $(-3, -4)$, $(6, -2)$. Find the lengths of its sides, and also of its diagonals.

8. Find the coordinates of the middle point of the st. line joining $(3, -2)$ and $(-3, 2)$.

9. Find the points of trisection of the st. line joining $(1, 3)$ and $(6, 1)$.

10. The st. line joining P $(-4, -3)$ and Q $(6, -1)$ is divided at R (x, y) so that PR : RQ = 5 : 2. Show that $x = -2y$.

11. Find the length of the st. line joining the origin to $(a, -b)$.

12. The st. line joining the origin to P $(-4, 7)$ is divided at R, Q so that OR : RQ : QP = 3 : 4 : 2. Find the distance RQ.

13. The length of a st. line is 17 and the coordinates of one end are $(-5, -8)$. If the ordinate of the other end is 7, find its abscissa.

14. Find in its simplest form the equation which expresses the fact that (x, y) is equidistant from $(5, 2)$ and $(3, 7)$.

15. Find the centre and radius of the circle which passes through $(5, 2)$, $(3, 7)$ and $(-2, 4)$.

16. Find the points which are distant 15 from $(-2, -10)$ and 13 from $(2, 14)$.

17. Prove that the vertices of a rt.-∠d △ are equidistant from the middle point of the hypotenuse.

Suggestion:—Take the vertex of the rt. ∠ for origin and the sides which contain the rt. ∠ for axes.

18. In any △ ABC prove that

$$AB^2 + AC^2 = 2\,(AD^2 + DC^2),$$

where D is the middle point of BC.

Suggestion:—Take D *as origin,* DC *as axis of* x *and the* ⊥ *to* BC *at* D *as axis of* y. *Let* DC $= a$, *and the coordinates of* A *be* (x_1, y_1).

19. If D is a point in the base BC of a △ ABC such that BD : DC $= m : n$, show that

$$n\,AB^2 + m\,AC^2 = (m+n)\,AD^2 + n\,BD^2 + m\,DC^2.$$

Suggestion:—Take D *as origin,* DC *as axis of* x *and the* ⊥ *to* BC *at* D *as axis of* y. *Let* BD $= -ma$, DC $= na$, *and the coordinates of* A *be* (x_1, y_1).

20. The vertices of a △ are the points (x_1, y_1), (x_2, y_2), (x_3, y_3). Find the coordinates of its centroid.

21. The st. line joining A $(2, 1)$ to B $(5, 9)$ is produced to C so that AC : BC $= 7 : 2$. Find the coordinates of C.

22. The st. line joining A $(3, -2)$ to B $(-4, -6)$ is produced to C so that AC : BC $= 3 : 2$. Find the coordinates of C.

THE AREA OF A TRIANGLE

17. To find the area of the △ of which the vertices are A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3).

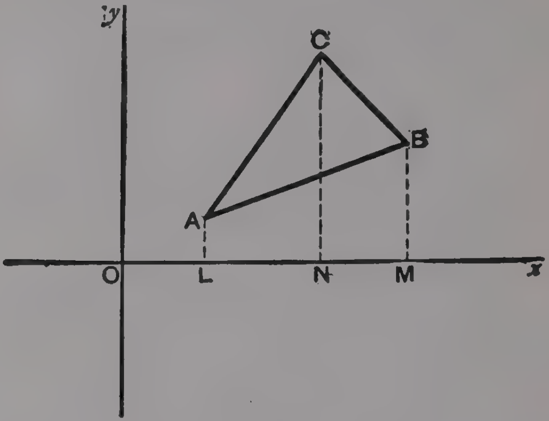


FIG. 11.

Draw the ordinates AL, BM, CN.

From the diagram,

△ ABC = ALNC + CNMB − ALMB.

The area of a quadrilateral of which two sides are ∥ = half the sum of the ∥ sides × the distance between the ∥ sides.

$$\therefore \text{ALNC} = \tfrac{1}{2}(\text{AL} + \text{CN}) \times \text{LN} = \tfrac{1}{2}(y_1 + y_3)(x_3 - x_1),$$

$$\text{CNMB} = \tfrac{1}{2}(\text{CN} + \text{BM}) \times \text{NM} = \tfrac{1}{2}(y_3 + y_2)(x_2 - x_3),$$

$$\text{ALMB} = \tfrac{1}{2}(\text{AL} + \text{BM}) \times \text{LM} = \tfrac{1}{2}(y_1 + y_2)(x_2 - x_1).$$

$$\therefore \triangle \text{ABC} = \tfrac{1}{2}\left\{(y_1 + y_3)(x_3 - x_1) + (y_3 + y_2)(x_2 - x_3) - (y_1 + y_2)(x_2 - x_1)\right\}.$$

Simplifying,

$$\triangle \text{ABC} = \tfrac{1}{2}\left\{\mathbf{x}_1(\mathbf{y}_2 - \mathbf{y}_3) + \mathbf{x}_2(\mathbf{y}_3 - \mathbf{y}_1) + \mathbf{x}_3(\mathbf{y}_1 - \mathbf{y}_2)\right\}.$$

NOTE.—*The points have been taken in circular order about the △ in the opposite direction to that in which the hands of a clock rotate; if they are taken in the same direction as the hands rotate, the formula will give the same result only it will appear to be negative; but, of course, the area of a △ must be positive.*

18. To find the area of the △ of which the vertices are (3, 2), (– 4, 3), (– 2, –4).

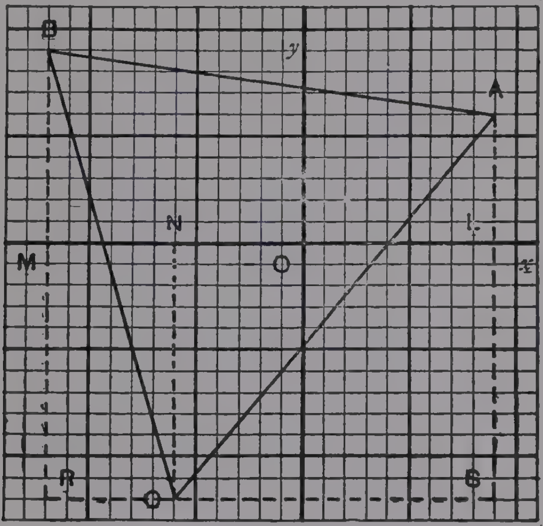


FIG. 12. (Unit = $\frac{3}{16}$ inch.)

Draw the diagram on squared paper. Draw the ordinates **AL, BM, CN.** Through **C** draw **RCS** ∥ **O**x to meet **AL, BM** produced at **S, R.**

$$\triangle \text{ABC} = \text{BRSA} - \triangle \text{BRC} - \triangle \text{ACS}.$$

$$\text{BRSA} = \tfrac{1}{2}(\text{BR} + \text{AS})\,\text{RS} = \tfrac{1}{2}(7+6)\times 7 = \frac{91}{2}.$$

$$\triangle \text{BRC} = \tfrac{1}{2}\,\text{BR} \times \text{RC} = \tfrac{1}{2} \times 7 \times 2 = \frac{14}{2}.$$

$$\triangle \text{ASC} = \tfrac{1}{2}\,\text{AS} \times \text{SC} = \tfrac{1}{2} \times 6 \times 5 = \frac{30}{2}.$$

$$\therefore \triangle \text{ABC} = \frac{91 - 14 - 30}{2} = \frac{47}{2}.$$

If we substitute the coordinates of A, B and C in the formula of § 17, we obtain

$$\triangle \text{ABC} = \tfrac{1}{2}\{3(3+4)+(-4)(-4-2)+(-2)(2-3)\}$$

$$= \tfrac{1}{2}(21+24+2) = \frac{47}{2};$$

the same result as before.

This illustrates the continuity of the symmetrical result found in § 17 for the area of a △.

19.—Exercises

1. Find, from a diagram, the area of the △ of which the vertices are (o, o), (a, b), (c, d). Check your result by using the formula of § 17.

2. Draw the following △s on squared paper and find their areas; checking your results by using the formula of § 17:—

(*a*) $(1, 4)$, $(-2, 2)$, $(5, -1)$;
(*b*) $(4, -2)$, $(-5, -1)$, $(-2, -6)$;
(*c*) $(0, 0)$, $(3, 4{\cdot}5)$, $(-2{\cdot}5, 4)$.

3. Find the area of the quadrilateral of which the vertices are $(3, 6)$, $(-2, 4)$, $(2, -2)$ and $(7, 3)$.

4. Find the area of the quadrilateral of which the vertices are $(0, 0)$, $(4, 0)$, $(3, 6)$ and $(-3, 3)$.

5. **D, E, F** are respectively the middle points of the sides **BC, CA, AB** of a △. Prove by the formula of § 17, taking **B** as origin and **BC** as axis of x, that △ **ABC** = 4 △ **DEF.**

6. Find the area of the △ of which the vertices are (x, y), $(3, 5)$, $(-2, 4)$; and thence show that if these points are in a st. line $5y - x = 22$.

7. Find the area of the △ **A** $(-3, 2)$, **B** $(7, 2)$, **C** $(3, 10)$; and show that the ⊥ from **A** to **BC** = **BC.**

8. A man starts from **O** and goes to **A,** from **A** to **B, B** to **C, C** to **D, D** to **O.** If **O** be taken as the origin and the coordinates of **A, B, C, D** are $(0, -3)$, $(8, 3)$, $(-4, 8)$, $(-4, 3)$, find the distance he has travelled, the unit being one mile.

9. Show from the formula for the area of a △ that **A** $(3, -2)$, **B** $(19, 10)$ and **C** $(7, 1)$ are in the same st. line. Find the ratio of **AC** to **CB.**

10. Show that if the coordinates of the vertices taken in order of a quadrilateral are (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4), its area is

$$\tfrac{1}{2}\{x_1(y_2 - y_4) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + x_4(y_1 - y_3)\}.$$

11. In the △ **OAB, P** is taken in **OA, Q** in **AB** and **R** in **BO** so that **OP : PA = AQ : QB = BR : RO** = 3 : 1. Show that △ **PQR** : △ **OAB** = 7 : 16.

Loci

20. The definition of a locus is:—

When a figure consisting of a line or lines contains all the points that satisfy a given condition, and no others, this figure is called the locus of these points.

The condition which the points satisfy may be expressed in the form of an equation involving the coordinates of the points. For example, take the locus of the points of which the ordinate is equal to 3. This condition, which is expressed by the equation $y = 3$ [*i.e.*:—$0x + y = 3$], is satisfied by an infinite number of points, as (0, 3), (1, 3), (2, 3), (7, 3), (−4, 3), etc. All such points are on a st. line AB ‖ to Ox and 3 units above it; and this st. line contains no points which do not satisfy the condition. Thus the equation $y = 3$ represents the line AB.

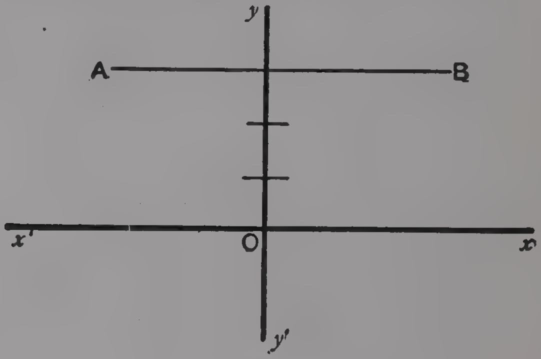


FIG. 13.

Similarly the equation $y = -3$ represents a st. line $\parallel$ Ox and three units below it; $x = 3$ represents a st. line $\parallel$ Oy and three units to the right of the origin, and $x = -5$ a st. line $\parallel$ Oy and 5 units to the left of the origin.

For another example let us take the condition to be that the abscissa and ordinate of each point are equal. The points (0, 0), (1, 1), (2, 2), (4, 4), $(-1, -1)$, $(-5, -5)$, etc., satisfy this condition. It is expressed by the equation $y = x$. If we draw a diagram on squared paper, mark some of these points on it and join them we get a st. line AB bisecting the $\angle$s xOy and x'Oy' every point on which satisfies the given condition. Between O and (1, 1) there are an infinite number of points, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{4}, \frac{1}{4})$, $(\frac{1}{10}, \frac{1}{10})$, $(\frac{9}{10}, \frac{9}{10})$, etc., which satisfy the condition, and so on continuously throughout the line. Thus the equation $y = x$ represents the line AB.

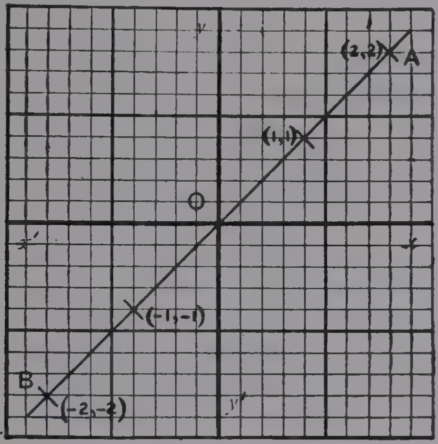


FIG. 14. (Unit = $\frac{4}{10}$ inch.)

Again, we may consider the point which moves so that its distance from the origin is always 5. Its locus is plainly the circumference of a circle. Particular points on this locus are (5, 0), (4, 3), (3, 4), (0, 5), (–3, 4), etc., and its equation is $\sqrt{x^2 + y^2} = 5$, or $x^2 + y^2 = 25$.

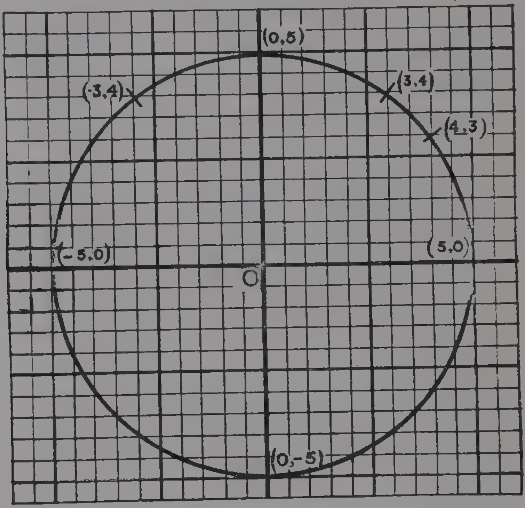


FIG. 15. (Unit = $\frac{2}{10}$ inch.)

21. In the equation of a locus the numbers that are the same for all points on the locus are called **constants**; while those that change in value continuously from point to point are called **variables.**

Thus, in the equation $x^2 + y^2 = 25$, x and y are variables and 25 is a constant.

22.—Exercises

1. Find four or five points on the locus represented by each of the following equations; and draw the locus on squared paper in each case:—

(*a*) $x = -4$; (*b*) $x + y = 0$; (*c*) $x - 2y = 0$;
(*d*) $3x + y = 0$; (*e*) $x = y + 4$; (*f*) $x^2 + y^2 = 169$.

2. A point moves so that its distance from the axis of x is 5 times its distance from the axis of y. Find the equation of its locus.

3. What locus is represented by the equation (*a*) $y = 0$; (*b*) $x = 0$?

4. A point moves so that it is equidistant from the origin and from (8, 0). Find the equation of its locus.

5. A point moves so that it is equidistant from the origin and from (3, −5). Find the equation of its locus, and draw the locus on squared paper.

6. A point is equidistant from (1, −2) and (−3, −4). Find the equation and draw the locus on squared paper.

7. A point moves so that its distance from (4, 3) is always 5. Find the equation and show that the locus passes through the origin.

8. The coordinates of the ends of the base of a △ are (−2, −3) and (4, −1), and the length of the median drawn to the base is 6. Find the equation of the locus of its vertex.

9. The coordinates of the ends of the base of a △ are (0, 0) and (5, 0), and its area is 10. Show that the equation of the locus of its vertex is $y = 4$.

10. The coordinates of the ends of the base of a △ are (−1, −2) and (5, 1) and its area is 9. Find the equation of the locus of its vertex.

23. An equation connecting two variables x and y has an infinite number of solutions. For example, in the equation $y = 3x + 7$, if any value is given to x, the corresponding value of y may then be determined. Thus, when

(*a*) $x = 0,\ y = 7$,
(*b*) $x = 1,\ y = 10$,
(*c*) $x = 2,\ y = 13$,
(*d*) $x = -1,\ y = 4$,
(*e*) $x = -3,\ y = -2$,
(*f*) $x = \frac{1}{3},\ y = 8$,
etc.

The, in general, continuous line which passes through all the points (*a*), (*b*), (*c*), etc., is the locus represented by this equation.

Another equation as $4x + 3y = 8$ has also an infinite number of solutions, and if these two equations are solved together, the common solution obtained, in this case $x = -1$, $y = 4$, gives the coordinates of the point of intersection of the loci represented by the equations.

Sets of solutions which satisfy the equation $4x + 3y = 8$ are given in the following table:—

x	y
(*d*) -1	4
(*g*) 2	0
(*h*) 5	-4

If we plot these two sets of results on squared paper, we see that the loci appear to be st. lines which intersect at the point (d) $(-1, 4)$.

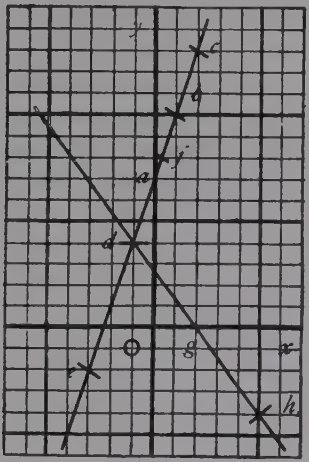


FIG. 16. (Unit = $\frac{1}{10}$ inch.)

24.—Exercises

1. Plot the following loci on squared paper and find the coordinates of their points of intersection :—

(a) $4x - y = 1$ and $x - 2y = -12$;
(b) $x + 2y = 7$ and $5x - 2y = 11$;
(c) $3x + 8y = -18$ and $4x + 3y = -1$;
(d) $3x + 4y = 0$ and $x^2 + y^2 = 100$;
(e) $3x - 5y + 45 = 0$ and $x^2 + y^2 = 169$.

2. Find the points where the locus $3x - 5y + 45 = 0$ cuts the axes.

3. Find the points where the locus $x^2 + y^2 = 6x$ cuts the axis of x.

4. Find the locus of a point such that the square of its distance from $(-a, o)$ is greater than the square of its distance from (a, o) by $2a^2$.

5. Find the equation of the locus of a point such that the square of its distance from $(-2, -1)$ is greater than the square of its distance from $(5, 3)$ by 11.

6. **A** $(1, 0)$ and **B** $(9, 0)$ are two fixed points and **P** is a variable point such that **PB** $= 3$ **PA.** Find the equation of the locus of **P.**

7. Plot the following loci and show that they are concurrent :—

$$3x + 4y = 10,\ 5x - 2y = 8,\ 4x + y = 9.$$

CHAPTER II

THE STRAIGHT LINE

25. To find the equation of a st. line in terms of the intercepts that it makes on the axes.

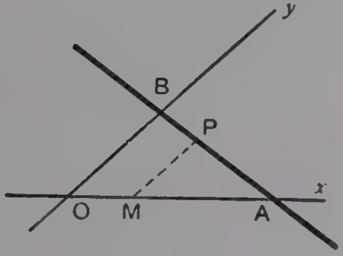


FIG. 17.

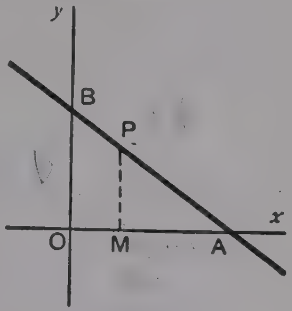


FIG. 18.

Let the st. line cut the axes at A, B so that OA = a, OB = b.

Take P (x, y) any point on the line, and draw PM ∥ Oy and terminated in Ox at M.

From the similar △s APM, ABO,

$$\frac{\text{PM}}{\text{BO}}=\frac{\text{AM}}{\text{AO}}.$$

$$\therefore \quad \frac{y}{b}=\frac{a-x}{a};$$

$$\text{or,} \quad \frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}=\mathbf{1}.$$

NOTE.—*It is seen from the diagrams that both the proof and the form of the equation are the same for oblique and rectangular axes.*

26. To find the equation of the st. line passing through A (x_1, y_1) and B (x_2, y_2).

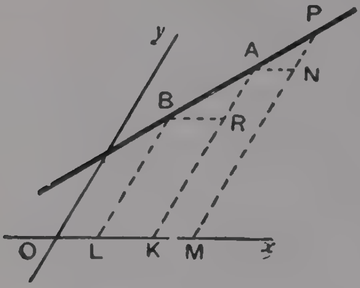


FIG. 19.

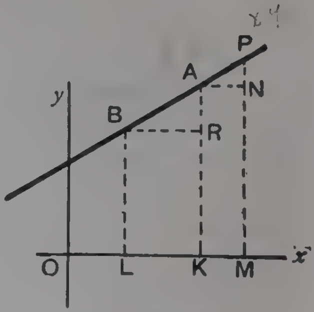


FIG. 20.

Take any point P (x, y) on the st. line.

Draw AK, BL PM ∥ Oy and terminated in Ox at K, L, M; and AN, BR ∥ Ox and respectively terminated in PM at N and AK at R.

From the similar △s PNA, ARB,

$$\frac{AN}{BR} = \frac{PN}{AR}.$$

$$AN = KM = OM - OK = x - x_1,$$

$$BR = LK = OK - OL = x_1 - x_2,$$

$$PN = PM - NM = PM - AK = y - y_1,$$

$$AR = AK - RK = AK - BL = y_1 - y_2$$

$$\therefore \frac{\mathbf{x} - \mathbf{x}_1}{\mathbf{x}_1 - \mathbf{x}_2} = \frac{\mathbf{y} - \mathbf{y}_1}{\mathbf{y}_1 - \mathbf{y}_2}.$$

NOTE.—*It is seen from the diagrams that both the proof and the form of the equation are the same for oblique and rectangular axes.*

27.—Exercises

1. The equation of the st. line passing through (4, 3) and (–2, 7) is by the formula of § 26

$$\frac{x-4}{4+2}=\frac{y-3}{3-7},$$

$$\text{or,}\ 2x+3y=17.$$

To find the intercepts which this line makes on the axes, let $y = 0$ and $\therefore\ x = 8\frac{1}{2}$, let $x = 0$ and $\therefore\ y = 5\frac{2}{3}$. By § 25 the equation of the line may now be written

$$\frac{x}{8\frac{1}{2}}+\frac{y}{5\frac{2}{3}}=1.$$

This is clearly the same as $2x + 3y = 17$.

2. Write down the equations of the st. lines which make the following intercepts on Ox, Oy respectively:—

(*a*) 5, 2; (*b*) –4, –6; (*c*) 3, –8.

3. Find the equations of the st. lines through the following pairs of points:—

(*a*) (6, 2), (3, 1); (*b*) (–1, 2), (–3, –7); (*c*) (4, –6), (–7, 2). Find the intercepts these st. lines make on the axes.

4. Find the point where the st. line which makes intercepts –3 and 5 on Ox and Oy respectively is cut by the st. line $x = -5$.

5. Find the point where the st. line making intercepts 7 and 2 on Ox and Oy respectively meets the st. line through (–2, 7) and (5, –3).

6. Find the point where the st. line through (3, 5) and (–7, –1) meets the st. line through (–8, 2) and (6, 5).

7. Prove that (11, 4) lies on the st. line joining (3, – 2) and (19, 10) and find the ratio of the segments into which the first point divides the join of the other two.

8. Find the equations of the sides of the △ of which the vertices are (4, – 2), (– 5, – 1), and (– 2, – 6). Find also the equations of the medians of the △ and the coordinates of its centroid.

9. The vertices of a quadrilateral are (3, 6), (– 2, 4), (2, – 2) and (7, 3). Find the equations of the four sides. Find also the equations of the two diagonals.

10. Find the vertices of the △ the sides of which are $11x - 3y = -45$, $5x - 11y = 47$ and $3x + 4y = 7$.

11. P (x_1, y_1) is any point and $\frac{x}{a} + \frac{y}{b} = 1$ cuts Ox, Oy at A, B respectively. Show that the area of the △ PAB =

$$\tfrac{1}{2}(bx_1 + ay_1 - ab).$$

28. As explained in elementary algebra, the **degree of a term,** with respect to certain letters, is the number of such letters that occur as factors in the term.

$3x$, $-5y$, ax, by are terms of the first degree with respect to x and y.

$5x^2$, $3y^2$, $-2xy$, ax^2 are terms of the second degree with respect to x and y.

29. **Degree of an equation.** An equation is said to be of the first degree in x and y when it contains a term, or terms, of the first degree in x and y, but no term of a higher degree than the first.

The **general equation of the first degree** in x and y is

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0.$$

An equation is said to be of the second degree in x and y when it contains a term, or terms, of the second degree in x and y, but no term of a higher degree than the second.

The **general equation of the second degree** in x and y is

$$\mathrm{A}x^2 + \mathrm{B}xy + \mathrm{C}y^2 + \mathrm{D}x + \mathrm{E}y + \mathrm{F} = 0,$$

or, in a more convenient form,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

30. To prove that an equation of the first degree always represents a st. line.

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be *any* three sets of simultaneous values of x and y which satisfy the equation $Ax + By + C = 0$.

$$\text{Then,}\quad (1)\ Ax_1 + By_1 + C = 0.$$
$$(2)\ Ax_2 + By_2 + C = 0.$$
$$(3)\ Ax_3 + By_3 + C = 0.$$

From (1) and (2),

$$\frac{A}{y_1 - y_2} = \frac{B}{x_2 - x_1} = \frac{C}{x_1y_2 - x_2y_1}.$$

Dividing the three terms of (3) respectively by these equal fractions and 0 by any one of them,

$$x_3(y_1 - y_2) + y_3(x_2 - x_1) + x_1y_2 - x_2y_1 = 0.$$

Rearranging the terms, we get

$$(4)\ x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0.$$

From § 17 the area of the △ formed by joining (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}$$

and ∴, from (4), in this case, the area of the △ is zero.

This can only be so when the three points are in a st. line, and ∴ as any three points the coordinates of which satisfy

$$Ax + By + C = 0$$

are in a st. line, this equation must always represent a st. line.

hange to this form

31. The equation $Ax + By + C = 0$ may be changed to the form

$$\frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1,$$

and by comparing this with the equation of § 25,

$$\frac{x}{a} + \frac{y}{b} = 1,$$

we see that the intercepts which the st. line $Ax + By + C = 0$ makes on the axes of x and y are respectively

$$-\frac{C}{A} \text{ and } -\frac{C}{B}.$$

The same results are obtained by alternately letting $y = 0$ and $x = 0$ in $Ax + By + C = 0$.

32. To obtain the result of § 26 from the general equation of the first degree.

Let (x_1, y_1), (x_2, y_2) be fixed points on the st. line represented by the general equation, and we have

(1) $Ax + By + C = 0$,

(2) $Ax_1 + By_1 + C = 0$,

(3) $Ax_2 + By_2 + C = 0$.

$A(x_1+x_2) + B(y_1+y_2) = -2C$

From (2) and (3),

$$(4) \quad \frac{A}{y_1 - y_2} = \frac{B}{x_2 - x_1} = \frac{C}{x_1y_2 - x_2y_1}.$$

∴, from (1) and (4),

$$x(y_1 - y_2) + y(x_2 - x_1) + x_1y_2 - x_2y_1 = 0.$$

This equation is seen to be the same as

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2},$$

when the latter is cleared of fractions and simplified.

33. To find the equation of a st. line in terms of its inclination to the axis of x and its intercept on the axis of y.

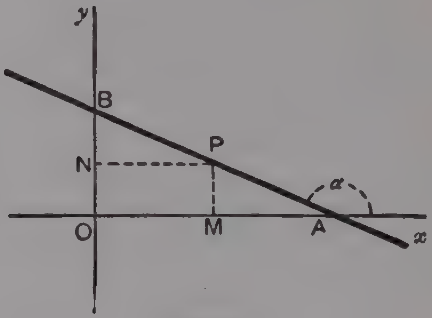


Fig. 21.

Let the st. line cut **O**x, **O**y at **A**, **B** respectively, $\angle$ **BA**$x = a$, and **OB** $= b$.

Take any point **P** (x, y) in the line, and draw **PM** $\perp$ **O**x, **PN** $\perp$ **O**y.

$$\textit{Tan}\ \mathrm{BPN} = \frac{\mathrm{BN}}{\mathrm{PN}} = \frac{\mathrm{BO} - \mathrm{PM}}{\mathrm{OM}} = \frac{b - y}{x}.$$

But, $\tan \mathrm{BPN} = \tan \mathrm{PAM} = -\tan a$.

$$\therefore -\tan a = \frac{b - y}{x},$$

$$\text{and } \therefore y = x \tan a + b.$$

If we let $\tan a = m$, the equation becomes

$$\mathbf{y = mx + b.}$$

In this equation m is called the **slope** of the line, and the $\angle\ a$, or $\tan^{-1}m$, is always measured by a rotation in the positive direction from the positive direction of $\mathrm{O}x$, *i.e.*, the $\angle$ is traced out by a radius vector starting from the position $\mathrm{A}x$ and rotating about A in the positive direction to the position AB.

NOTE.—*For oblique axes the proof and result are different from those given above for rectangular axes.*

34. The equation $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ may be changed to

$$y = -\frac{\mathrm{A}}{\mathrm{B}}x - \frac{\mathrm{C}}{\mathrm{B}},$$

from which by comparison with

$$y = mx + b,$$

it is seen that the slope of the st. line $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ is $-\frac{\mathrm{A}}{\mathrm{B}}$, and its intercept on the axis of y is $-\frac{\mathrm{C}}{\mathrm{B}}$.

35. To find the equation of a st. line in terms of the ⊥ on it from the origin and the ∠ made by a positive rotation from Ox to this ⊥.

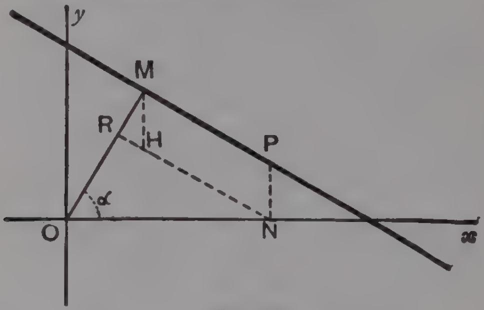


FIG. 22.

Let the ⊥ OM from O to the line $= p$, and ∠ xOM $= a$.

Take any point P (x, y) in the st. line.

Draw PN ⊥ Ox, NR ⊥ OM, MH ∥ Oy to meet NR at H.

$$\text{OR} + \text{RM} = p.$$

$$\text{OR} = \text{ON} \cos \text{RON} = x \cos a.$$

$$\text{RM} = \text{MH} \cos \text{RMH} = \text{PN} \cos \text{MO}y = y \sin a.$$

$$\therefore \mathbf{x \cos a + y \sin a = p}.$$

NOTE.—*For oblique axes the proof and result are different from those given above for rectangular axes.*

36. To reduce the equation $\text{A}x + \text{B}y + \text{C} = 0$ to the form $x \cos a + y \sin a = p$, where p is always a positive quantity.

The equations

$$x \cos a + y \sin a - p = 0$$
$$\text{A}x + \text{B}y + \text{C} = 0$$

will be identical if

$$\frac{\cos a}{\text{A}} = \frac{\sin a}{\text{B}} = \frac{-p}{\text{C}}.$$

If C is a positive quantity,

$$\frac{p}{C} = \frac{\cos a}{-A} = \frac{\sin a}{-B} = \frac{\sqrt{\cos^2 a + \sin^2 a}}{\sqrt{A^2 + B^2}} = \frac{1}{\sqrt{A^2 + B^2}}.$$

$$\therefore \cos a = \frac{-A}{\sqrt{A^2 + B^2}},\ \sin a = \frac{-B}{\sqrt{A^2 + B^2}},\ \text{and } p = \frac{C}{\sqrt{A^2 + B^2}}.$$

If C is a negative quantity, these results should be written

$$\cos a = \frac{A}{\sqrt{A^2 + B^2}},\ \sin a = \frac{B}{\sqrt{A^2 + B^2}},\ p = \frac{-C}{\sqrt{A^2 + B^2}}.$$

Thus the equation is

$$\frac{\mp Ax}{\sqrt{A^2 + B^2}} + \frac{\mp By}{\sqrt{A^2 + B^2}} = \frac{\pm C}{\sqrt{A^2 + B^2}};$$

the upper signs being taken when C represents a positive quantity and the lower signs when C represents a negative quantity.

37. Ex. 1. Reduce the equation $3x + 4y - 12 = 0$ to the form $x \cos a + y \sin a = p$.

Here $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

Dividing the given equation by 5

$$\frac{3}{5}x + \frac{4}{5}y = \frac{12}{5}.$$

$\cos a = \frac{3}{5}$, $\sin a = \frac{4}{5}$, and $\therefore a = \tan^{-1}\left(\frac{4}{3}\right)$ while the $\perp$ from the origin on the line is $\frac{12}{5}$.

Ex. 2. Reduce the equation $x - y + 7 = 0$ to the form $x \cos a + y \sin a = p$.

Here $\sqrt{1^2+1^2} = \sqrt{2}$.

Dividing the given equation by $-\sqrt{2}$

$$-\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$

$$\text{or, } x \cos 135° + y \sin 135° = \frac{7}{\sqrt{2}}.$$

i.e., $a = 135°$, and the ⊥ from the origin on the st. line is $\frac{7}{\sqrt{2}}$.

38. **To find the equation of a st. line in terms of the coordinates of a fixed point on the line and the ∠ which the line makes with Ox.**

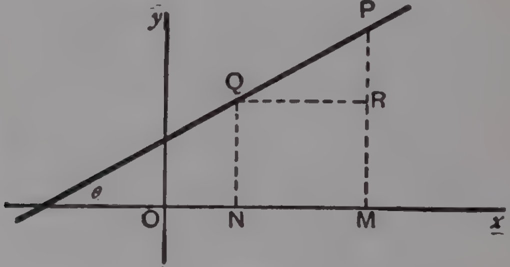


Fig. 23.

Let Q (x_1, y_1) be the fixed point and θ the ∠.

Take any point P (x, y) on the line and let QP $= r$. Draw PM, QN ⊥ Ox and QR ⊥ PM.

$$QR = PQ \cos PQR.$$

$$QR = NM = x - x_1, \text{ and } \angle PQR = \angle \theta.$$

$$\therefore \quad \frac{x - x_1}{\cos\theta} = r.$$

$$PR = PQ \sin PQR.$$

$$PR = PM - RM = PM - QN = y - y_1.$$

$$\therefore \quad \frac{y - y_1}{\sin\theta} = r.$$

$$\therefore \quad \frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r.$$

This form will frequently be found useful in problems that involve the distance between two points on a st. line.

If $\cos\theta = l$ and $\sin\theta = m$, the equation becomes:—

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r.$$

l and m are called the **direction cosines** of the st. line, and $l^2 + m^2 = 1$.

39. For convenience of reference the different forms of the equation of the st. line are here collected:—

$$(1)\quad Ax + By + C = 0.$$

$$(2)\quad \frac{x}{a} + \frac{y}{b} = 1.$$

$$(3)\quad \frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}.$$

$$(4)\quad y = mx + b.$$

$$(5)\quad x\cos\alpha + y\sin\alpha = p.$$

$$(6)\quad \frac{x - x_1}{l} = \frac{y - y_1}{m} = r.$$

If the st. line passes through the origin (4) takes the convenient form:—

$$(7)\quad y = mx.$$

If the st. line joins the origin to a fixed point (x_1, y_1), we get, by letting $x_2 = y_2 = 0$ in (3):—

$$(8)\ \frac{x}{x_1} = \frac{y}{y_1}.$$

If the st. line passes through (x_1, y_1) and its slope is m, the equation is easily seen from **(4) to be**

$$(9)\ y - y_1 = m\,(x - x_1).$$

40.—**Exercises**

1. Name the constants and variables in each of the nine equations of § 39. Explain the meaning of each constant. Which of these equations are of the same form for rectangular and oblique axes?

2. Draw the following st. lines on squared paper:—

(*a*) $x + 2y = 8$; (*b*) $3x - 7y = -5$; (*c*) $\frac{x}{3} + \frac{y}{5} = -1$; (*d*) $2x + 3y = 13$; (*e*) $3y = 4x$.

3. Find the equation of the st. line.

(*a*) through the origin and making an ∠ of 30° with $\mathrm{O}x$;

(*b*) through the origin and making an ∠ of 120° with $\mathrm{O}x$;

(*c*) through (0, 5) and making the ∠ $\tan^{-1}\frac{5}{7}$ with $\mathrm{O}x$;

(*d*) through (0, – 3) and making the ∠ $\cos^{-1}\frac{3}{5}$ with $\mathrm{O}x$;

(*e*) through (– 3, – 4) and making the ∠ 45° with $\mathrm{O}x$.

4. In the △ of which the vertices are (– 2, 5), (3, – 7), (4, 2)

(*a*) find the slope of each side;

(*b*) show that the medians are concurrent and find the centroid.

5. **O** (0, 0), **A** (6, 0), **B** (4, 6), **C** (2, 8) are the vertices of a quadrilateral. Show that the st. lines joining the

middle points of OA, BC, of AB, CO and of OB, AC are concurrent, and find the coordinates of their common point.

6. Find the equation to the st. line through $(-4, 3)$ that cuts off equal intercepts from the axes.

7. Find the length of the ⊥ from the origin to the line $3x + 7y = 10$; find the ∠ which this ⊥ makes with Ox.

8. What is the condition that the st. line $Ax + By + C = 0$ may

(*a*) pass through the origin;

(*b*) be ∥ Ox;

(*c*) be ∥ Oy;

(*d*) cut off equal intercepts from the axes;

(*e*) make ∠ 45° with Ox;

9. What must be the value of m if the line $y = mx + 7$ passes through $(-2, 5)$?

10. Find the values of m and b, if the st. line $y = mx + b$ passes through $(-2, 3)$ and $(7, 2)$.

11. Find the values of a and b, if the st. line $\frac{x}{a} + \frac{y}{b} = 1$ passes through $(-2, -5)$ and $(4, -2)$.

12. Show that the points $(4a, -3b)$, $(2a, 0)$, $(0, 3b)$ are in a st. line.

13. Show that the intercept made on the line $x = k$ by the lines $Ax + By + C = 0$ and $Ax + By + C' = 0$ is the same for all values of k.

14. $P(x_1, y_1)$ is any point and the line $Ax + By + C = 0$ cuts Ox, Oy at N, R respectively. Show that △ PNR =

$$\frac{C}{2\,AB}(Ax_1 + By_1 + C).$$

THE ANGLE BETWEEN TWO STRAIGHT LINES

41. To find the ∠ between two st. lines whose equations are given.

(i) Let the given equations be $y = m_1x + b_1$ and $y = m_2x + b_2$.

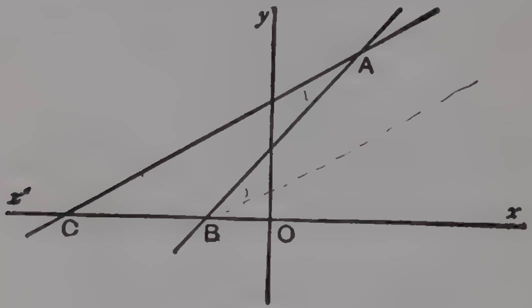


FIG. 24.

Let **AB** be the line $y = m_1x + b_1$ and **AC** be the line $y = m_2x + b_2$ when **B**, **C** are on the axis of x. Let $\angle$ **BAC** $= \theta$.

Then $m_1 = \tan \text{AB}x$, $m_2 = \tan \text{AC}x$. ✓

$$\angle\, \theta = \angle\, \text{AB}x - \angle\, \text{AC}x.$$ (See dotted ‖ line)

$$\therefore\ \tan\theta = \frac{\tan \text{AB}x - \tan \text{AC}x}{1 + \tan \text{AB}x \,.\, \tan \text{AC}x} = \frac{m_1 - m_2}{1 + m_1m_2}.$$

$$\therefore\qquad \theta = \tan^{-1}\frac{m_1 - m_2}{1 + m_1m_2}.$$ (? $\theta = \tan^{-1}\theta$) see page iv of [illegible]

(ii) Let the given equations be $\text{A}x + \text{B}y + \text{C} = 0$ and $\text{A}_1x + \text{B}_1y + \text{C}_1 = 0$.

These equations may be changed to

$$y = -\frac{\text{A}}{\text{B}}x - \frac{\text{C}}{\text{B}} \text{ and } y = -\frac{\text{A}_1}{\text{B}_1}x - \frac{\text{C}_1}{\text{B}_1}.$$

∴, writing $-\frac{A}{B}$ for m_1 and $-\frac{A_1}{B_1}$ for m_2 in the above result, the ∠ between the lines

$$= \tan^{-1}\frac{-\frac{A}{B}+\frac{A_1}{B_1}}{1+\frac{AA_1}{BB_1}} = \tan^{-1}\frac{A_1B - AB_1}{AA_1 + BB_1}.$$

42. Condition of Parallelism. If two st. lines are ∥, they make equal ∠s with the axis of x, *i.e.*, their slopes are the same.

∴, if their equations are $y = m_1x + b_1$ and $y = m_2x + b_2$, the condition is

$$\mathbf{m_1 = m_2}.$$

If their equations are $Ax + By + C = 0$ and $A_1x + B_1y + C_1 = 0$, the condition is

$$-\frac{A}{B} = -\frac{A_1}{B_1},$$

$$\text{or, } AB_1 - A_1B = 0.$$

This may also be written $\frac{A}{A_1} = \frac{B}{B_1}$; and we see that the equation $ax + by = k$ can be made to represent an infinite number of ∥ st. lines by giving different values to k; as:— $ax + by = k_1$, $ax + by = k_2$, etc.

43. Condition of Perpendicularity. If the st. lines $y = m_1x + b_1$, $y = m_2x + b_2$ are ⊥,

$$\tan^{-1}\frac{m_1 - m_2}{1 + m_1m_2} = \frac{\pi}{2}.$$

$$\therefore \quad \frac{m_1 - m_2}{1 + m_1m_2} = \infty.$$

This will be true if $1 + m_1m_2 = 0$, and ∴ the required condition is

$$m_1m_2 = -1.$$

Similarly, if $Ax + By + C = 0$ and $A_1x + B_1y + C_1 = 0$ are ⊥.

$$AA_1 + BB_1 = 0.$$

The st. lines

$$Ax + By + C = 0$$

$$Bx - Ay + C_1 = 0$$

satisfy the above condition and ∴ are ⊥ to each other.

44.—Exercises

1. Find the ∠ between the st. lines

(*a*) $2x - 3y = 9$ and $x + 5y = 11$;

(*b*) $3x + 5y = 12$ and $(17\sqrt{3} + 30)x + 33y = 19$;

(*c*) $5y = 3x + 12$ and $5x + 3y = 17$;

(*d*) $4x + 7y = 13$ and $3x - y = 6$.

2. Find the equation of the st. line ∥ to $6x - 7y = 13$ and passing through $(-2, -5)$.

Solution:—The required equation is

$$6(x + 2) - 7(y + 5) = 0;$$

i.e., $6x - 7y = 23$.

3. Find the equation of a st. line through $(-3, -5)$ and ∥ to $9x + 4y = 18$.

4. Find the equation of the st. line drawn through $(-2, -5)$ and ⊥ to $6x - 7y = 13$.

Solution:—The required equation is

$$7(x + 2) + 6(y + 5) = 0;$$

i.e., $7x + 6y + 44 = 0$.

5. Find the equation of the st. line drawn through $(4, 2)$ and ⊥ $3x - 2y = 1$.

6. Find the equation of the st. line passing through $(-3, 5)$ and || to the st. line joining $(2, 6)$ and $(7, -1)$.

7. Find the equation of the st. line passing through $(2, 6)$ and ⊥ the st. line joining $(-3, 5)$ and $(7, -1)$.

8. Find the equations of st. lines drawn through $(5, 7)$ which make ∠s 45° and 135° with Ox.

9. Find the equations of st. lines drawn through $(-5, -3)$ which make ∠s 30° and 150° with Ox.

10. Show that the ⊥s from the vertices of the △ $(1, -3)$, $(-5, -2)$, $(4, 7)$ to the opposite sides are concurrent; and find the coordinates of the orthocentre.

11. Show that the ⊥s from the vertices of the △ $(0, 0)$, $(a, 0)$, (b, c) to the opposite sides are concurrent; and find the orthocentre.

12. Find the ratio into which the ⊥ from the origin on the st. line joining $(2, 6)$ and $(5, 1)$ divides the distance between these points.

13. Find the equations of the st. lines which pass through (h, k) and form with $y = mx + b$ an isosceles △ of which the vertex is at the given point and each base $\angle = a$.

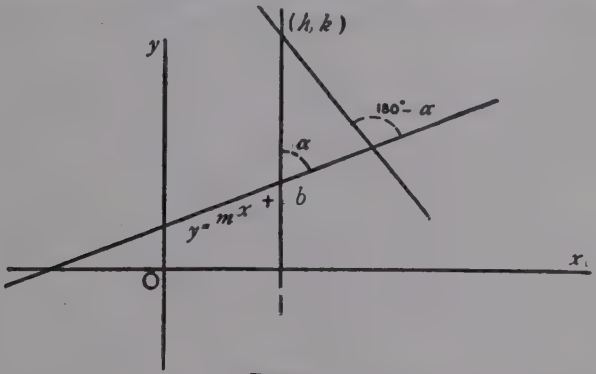


Fig. 25.

Solution:—Let $y - k = \mathsf{M}\ (x - h)$ represent one side of the △ where the value of M is to be found.

$$\tan a = \frac{M - m}{1 + mM}.$$

$$\therefore \quad M = \frac{m + \tan a}{1 - m \tan a}.$$

$\therefore$ the equation of this side is

$$y - k = \frac{m + \tan a}{1 - m \tan a}(x - h).$$

If for a we substitute $108° - a$, the equation of the other side is found to be

$$y - k = \frac{m - \tan a}{1 + m \tan a}(x - h).$$

14. Find the equations of the st. lines passing through (2, 8) and making an ∠ of 30° with $3x - 12y = 7$.

15. Find the equations of the st. lines passing through $(-1, -2)$ and making an ∠ of 45° with $\frac{x}{7} + \frac{y}{5} = 1$.

16. Show that the equation of the st. line through (a, b) and making an ∠ of 60° with $x \cos a + y \sin a = p$ is $y - b = (x - a) \tan (a \pm 30°)$.

17. Show that the right bisectors of the sides of the △ $(0, 0)$, $(a, 0)$, (b, c) are concurrent; and find their point of intersection.

18. Find the equation of a st. line ⊥ to $Ax + By + C = 0$, and at a distance p from the origin.

PERPENDICULARS

45. To find the length of the $\perp$ from P (x_1, y_1) to $Ax + By + C = 0$.

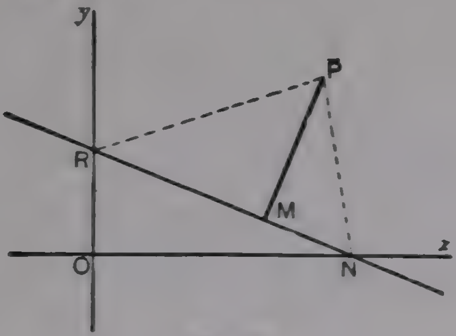


FIG. 26.

Draw PM $\perp$ the given st. line. Join P to N, R the points where the given st. line cuts Ox, Oy.

$$\triangle PRN = \tfrac{1}{2} PM \,.\, RN.$$

$$ON = -\frac{C}{A} \text{ and } OR = -\frac{C}{B},$$

$$\therefore RN = \sqrt{\frac{C^2}{B^2} + \frac{C^2}{A^2}} = \frac{C}{AB}\sqrt{A^2 + B^2}.$$

By the formula of § 17,

$$\angle PRN = \tfrac{1}{2}\left\{x_1\left(-\frac{C}{B}\right) - \frac{C}{A}\left(y_1 + \frac{C}{B}\right)\right\}$$

$$= -\frac{C}{2AB}(Ax_1 + By_1 + C).$$

$$\therefore \tfrac{1}{2} PM \,.\, \frac{C}{AB}\sqrt{A^2 + B^2} = -\frac{C}{2AB}(Ax_1 + By_1 + C).$$

$$\therefore PM = -\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}.$$

The length of the $\perp$ is $\therefore$

$$\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}.$$

In the diagram **AB** is the line represented by the equation $4x + 5y - 20 = 0$.

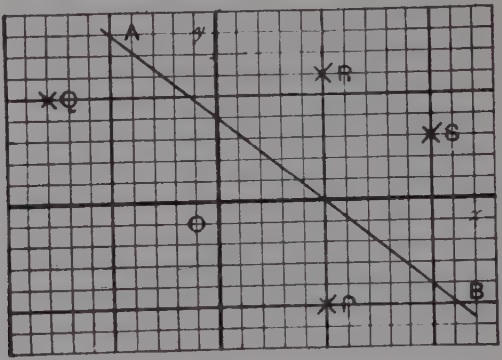


FIG. 27. (Unit = $\frac{1}{10}$ inch.)

If in the *expression* $4x + 5y - 20$ we substitute the coordinates of points **O, P, Q, R, S** which are not in the line, the following results are obtained.

For **O** $(0, 0)$, $4x + 5y - 20 = -20$.
" **P** $(5, -5)$, $4x + 5y - 20 = -25$.
" **Q** $(-8, 5)$, $4x + 5y - 20 = -27$.
" **R** $(5, 6)$, $4x + 5y - 20 = +30$.
" **S** $(10, 3)$, $4x + 5y - 20 = +35$.

In these results it will be observed that:—

For the origin the sign of the value of the expression is the same as the sign of the absolute term.

For other points that lie on the same side of the given st. line as the origin the signs of the values of the expression are the same as the sign of the result for the origin; while, for points on the side remote from the origin the signs of the values of the expression are different from the sign of the result for the origin.

A formal proof of these properties is given in the next article.

46. To prove that the sign of the expression $Ax + By + C$ is different for points on opposite sides of the line $Ax + By + C = 0$.

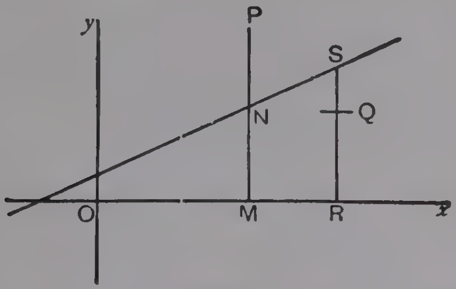


FIG. 28.

P (x_1, y_1), Q (x_2, y_2) are any points on opposite sides of $Ax + By + C = 0$.

Draw PM, QR $\perp$ Ox and let them cut the given line at N, S.

$$y_1 = PM = PN + NM; \; y_2 = QR = SR - SQ.$$

$$\therefore \; Ax_1 + By_1 + C = Ax_1 + B.PN + B.NM + C,$$

and $\quad Ax_2 + By_2 + C = Ax_2 + B.SR - B.SQ + C.$

But, $\because$ N and S are both on the given line,

$$Ax_1 + B.NM + C = 0,$$

and $\quad Ax_2 + B.SR + C = 0.$

$$\therefore \quad Ax_1 + By_1 + C = B.PN,$$

and $\quad Ax_2 + By_2 + C = - B.SQ.$

$\therefore$, since PN, SQ are both taken as positive quantities, $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ have opposite signs.

When $x = 0$ and $y = 0$ the expression $\mathrm{A}x + \mathrm{B}y + \mathrm{C}$ becomes C, ∴ a point whose coordinates when substituted in $\mathrm{A}x + \mathrm{B}y + \mathrm{C}$ gives the same sign as C is on the same side of the st. line $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ as the origin.

47. **Sign of the Perpendicular.** It follows from the preceding article that, if the positive sign is always taken for $\sqrt{\mathrm{A}^2 + \mathrm{B}^2}$, when the sign of

$$\frac{\mathrm{A}x_1 + \mathrm{B}y_1 + \mathrm{C}}{\sqrt{\mathrm{A}^2 + \mathrm{B}^2}}$$

is the same as the sign of C, the point (x_1, y_1) and the origin are on the same side of the line $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$; and when the sign of this fraction is different from that of C, the point (x_1, y_1) and the origin are on opposite sides of $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$.

48. **To find the equations of the bisectors of the ∠s between the lines $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ and $\mathrm{A}_1x + \mathrm{B}_1y + \mathrm{C}_1 = 0$.**

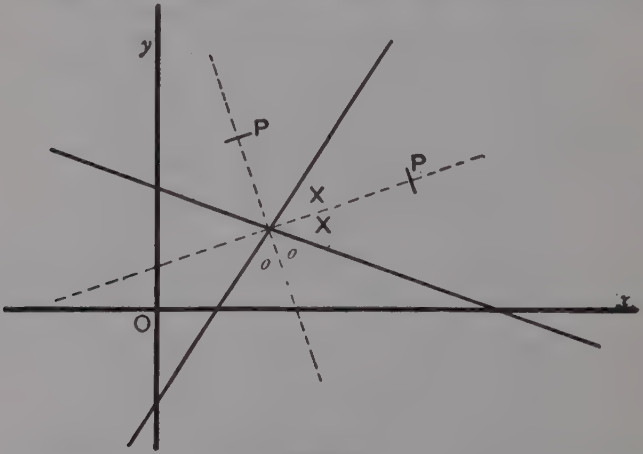


FIG. 29.

The ⊥s to the st. lines from any point P (x, y) on either bisector are equal to each other,

∴ the required equations are

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \pm \frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}}.$$

If the equations are so written that C and C_1 have the same sign and P is on the bisector of the ∠ that contains the origin, the ⊥s from P have the same sign as the ⊥s from the origin on the lines, and the equation of the bisector is

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}}.$$

If P is on the bisector of the ∠ which does not contain the origin, the ⊥s from P have opposite signs and the equation of the bisector is

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = -\frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}}.$$

49. To find the distance from (a, b) to $Ax+By+C=0$ in the direction whose direction cosines are l, m.

The equation of the st. line passing through (a, b) in the given direction is, by § 38,

$$\frac{x-a}{l} = \frac{y-b}{m} = r.$$

$$\therefore\ x = a + lr,\ y = b + mr.$$

Substituting these values for x and y in $Ax + By + C = 0$,

$$Aa + Alr + Bb + Bmr + C = 0.$$

$$\therefore\ r = -\frac{Aa + Bb + C}{Al + Bm}.$$

50. The length of the ⊥ from (a, b) to $Ax + By + C = 0$ may be deduced from the result of § 49.

For, if $\dfrac{x-a}{l} = \dfrac{y-b}{m}$ and $Ax + By + C = 0$ are ⊥ to each other,

$$\frac{A}{l} = \frac{B}{m} = \frac{Al + Bm}{l^2 + m^2} = Al + Bm,$$

since $l^2 + m^2 = 1$.

Also each of these fractions

$$= \frac{\sqrt{A^2 + B^2}}{\sqrt{l^2 + m^2}} = \sqrt{A^2 + B^2}.$$

$$\therefore\ Al + Bm = \sqrt{A^2 + B^2},$$

and the length of the ⊥ is

$$\frac{Aa + Bb + C}{\sqrt{A^2 + B^2}}.$$

51. To find the equation of a line passing through the intersection of two loci.

The equation

$$k\,(\mathrm{A}x + \mathrm{B}y + \mathrm{C}) + l\,(\mathrm{A}_1x + \mathrm{B}_1y + \mathrm{C}_1) = 0, \qquad (1)$$

being of the first degree in x and y represents a st. line.

If (x_1, y_1) is the point of intersection of

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0 \qquad (2)$$

$$\text{and } \mathrm{A}_1x + \mathrm{B}_1y + \mathrm{C}_1 = 0, \qquad (3)$$

the values x_1, y_1 substituted for x, y will plainly satisfy equation (1), and ∴ the st. line (1) must pass through the point of intersection of the st. lines (2) and (3).

From the same reasoning the following more general theorem is seen to be true:—

If two equations are multiplied by any numbers and the results either added or subtracted, the resulting equation represents a locus that passes through the point (or points) of intersection of the loci represented by the first two.

52. *Example*—Find the equation of the st. line passing through the intersections of $17x - 7y = 9$, $3x + 19y = 34$ and ⊥ to $11x - 4y = 13$.

$$17x - 7y - 9 + l\,(3x + 19y - 34) = 0$$

is a st. line passing through the intersection of the first two lines.

This equation may be written

$$(3l + 17)\,x + (19l - 7)\,y - 34l - 9 = 0.$$

If this line is ⊥ to $11x - 4y = 13$,

$$11\,(3l + 17) - 4\,(19l - 7) = 0$$

$$\therefore\ l = 5,$$

and the required equation is found to be

$$32x + 88y = 179.$$

53. To find the condition that the three st. lines

$$a_1 x + b_1 y + c_1 = 0 \ldots\ldots\ldots\ldots (1)$$

$$a_2 x + b_2 y + c_2 = 0 \ldots\ldots\ldots\ldots (2)$$

$$a_3 x + b_3 y + c_3 = 0 \ldots\ldots\ldots\ldots (3)$$

may be concurrent.

If the three st. lines are concurrent, the coordinates of the common point satisfy the three equations.

For that point, from (1) and (2),

$$\frac{x}{b_1 c_2 - b_2 c_1} = \frac{y}{c_1 a_2 - c_2 a_1} = \frac{1}{a_1 b_2 - a_2 b_1}.$$

Dividing the terms of (3) respectively by these equal fractions,

$$a_3 (b_1 c_2 - b_2 c_1) + b_3 (c_1 a_2 - c_2 a_1) + c_3 (a_1 b_2 - a_2 b_1) = 0.$$

This is the relationship that must hold among the constants in order that the lines may be concurrent.

TRANSFORMATION OF COORDINATES

54. It is often necessary to change the coordinates involved in a problem into a different set which are referred to axes drawn

(*a*) from a new origin, or

(*b*) in directions different from the original axes.

55. **To change from a pair of axes to another pair which are ∥ to the former, but have a different origin.**

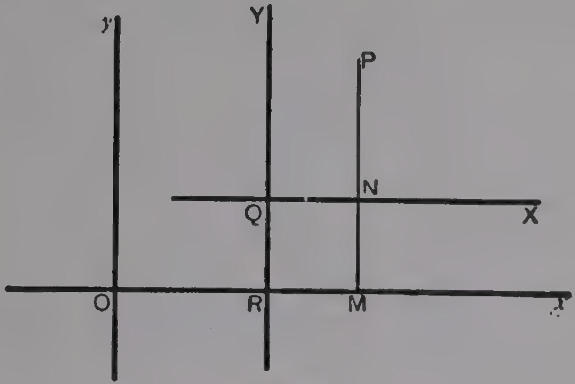


FIG. 30.

Q (h, k) is the new origin.

2, 3

Let P (x, y) be any point referred to Ox and Oy and X, Y the coordinates of the same point referred to the new axes QX and QY.

8, 6

Draw PNM ⊥ to Ox and QX, and let YQ cut Ox at R.

8 = 8 = 2 + 6 = 2 + 6

$$x = \text{OM} = \text{OR} + \text{QN} = h + \text{X}.$$

$$y = \text{PM} = \text{QR} + \text{PN} = k + \text{Y}.$$

6 = 3 + 3 = 3 + 3

Thus, if for x, y respectively we substitute $h + \text{X}$, $k + \text{Y}$ in any equation the origin is changed to the point (h, k).

To return to the original origin the substitutions would be $\text{X} = x - h$, $\text{Y} = y - k$.

6 = 8 − 2 3 = 6 − 3

56. **To change the direction of the axes, without changing the origin, the axes being rectangular.**

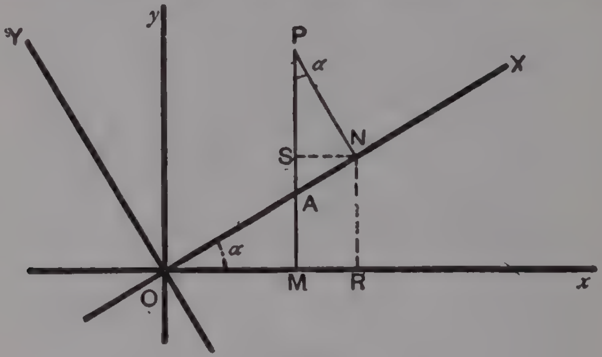


Fig. 31.

Let P (x, y) be any point referred to Ox, Oy; and X, Y the coordinates of the same point referred to axes OX, OY such that ∠ XOx = a.

Draw PM ⊥ Ox, PN ⊥ OX, NR ⊥ Ox, NS ⊥ PM.

$$\angle \text{NPS} = 90° - \angle \text{NAP} = 90° - \angle \text{MAO} = a.$$

$$x = \text{OM} = \text{OR} - \text{NS} = \text{X} \cos a - \text{Y} \sin a.$$

$$y = \text{PM} = \text{NR} + \text{PS} = \text{X} \sin a + \text{Y} \cos a.$$

Thus, if for x, y we substitute respectively $\text{X} \cos a - \text{Y} \sin a$, $\text{X} \sin a + \text{Y} \cos a$, the axes are rotated in the positive direction through an ∠ a.

57. Exercises

(*a*)

1. Find the length of the $\perp$

(*a*) from $(-4, 7)$ to $5x - 2y = 4$; ✓

(*b*) from $(-4, -3)$ to $\frac{x}{2} + \frac{y}{3} = 1$;

(*c*) from $(3, -2)$ to $y = 7x + 1$;

(*d*) from $(-2, -7)$ to the st. line joining $(5, 3)$ and $(-3, -7)$; ✓

(*e*) from the origin to the st. line joining $(7, 0)$ and $(0, -5)$.

2. Find the distance between the $\parallel$ lines $4x - 3y = 9$, $4x - 3y = 2$. ✓

3. Find the distance between the $\parallel$ lines $ax + by + c_1 = 0$, $ax + by + c_2 = 0$.

4. Find the point in the line $\frac{x}{2} + \frac{y}{5} = -1$ such that its $\perp$ distance from the st. line joining $(2, 7)$, $(5, 3)$ is 8.

5. Find the equation of the st. line through the intersection of $3x - 2y = 12$, $5x + 4y = 9$ and $\parallel$ to $\frac{x}{3} + \frac{y}{4} = 1$. (See §§ 51 and 52).

6. Find the equation of the st. line joining the origin to the intersection of $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$.

7. Find the distance from the point of intersection of $7x - 5y = 13$, $4x + 9y = 43$ to the line $12x = 5y$.

8. Find the equation of the st. line passing through the intersection of $y = mx + c$, $y = m_1x + c_1$ and $\perp$ to $\frac{x}{a} + \frac{y}{b} = 1$.

9. Show that the st. lines $x + 2y = 5$, $2x + 3y = 8$, $3x + y = 5$, $x + y = 3$ and $2x - y = 0$ are concurrent.

10. Find the condition that the lines $ax + hy + g = 0$, $hx + by + f = 0$, $gx + fy + c = 0$ are concurrent.

11. Find the equation of the st. line passing through the intersection of $y = mx + c$, $y = m_1x + c_1$ and also through (a, b).

12. Find the equation of the st. line joining the origin to the point of intersection of $\mathbf{A}x + \mathbf{B}y + \mathbf{C} = 0$ and $\mathbf{A}_1x + \mathbf{B}_1y + \mathbf{C}_1 = 0$.

13. Find the distance from the orthocentre of the △ **O** (0, 0), **A** (8, 0), **B** (3, 5) to the st. line **AB**.

14. Plot the lines $2x - 3y = 1$, $3x + y = 7$ on squared paper and find the intercepts that the bisectors of the ∠s between them make on the axis of y.

15. Find the equations of the bisectors of the ∠s between $5x - 12y = 17$ and $8x + 15y = 31$.

16. Show that the bisectors of the supplementary ∠s between $y = mx + a$ and $y = m_1x + a_1$ are ⊥ to each other.

17. Find the equations of the bisectors of the ∠s between the st. lines joining (4, 5) and (–5, 2) respectively to (3, –7).

18. Show that the st. lines $x + 6y = 15$, $2x - 5y + 4 = 0$ and $9x + y = 29$ are concurrent.

19. The sides of a △ are $3x + 4y = 15$, $12x - 5y = 17$, $24x + 7y = 30$. Plot the lines on squared paper and find the point where the bisectors of the interior ∠s of the △ intersect.

20. Find the equation of the st. line passing through the intersection of the lines $2x - 3y = 6$ and $3x + 4y = 18$, and also through the middle point of the st. line joining $(-1, 2)$ and $(3, 4)$.

21. Find the distance from $(5, 3)$ in the direction in which the slope is $\frac{1}{\sqrt{3}}$ to the line $7x - 11y = 13$.

22. Find the distance from $(-4, 6)$ in the direction of which the slope is 1 to the line $\frac{x}{2} + \frac{y}{3} = 1$.

Draw the diagram on squared paper.

23. The sum of the distances from a point to the lines $x + 2y = 7$, $5x - 2y = 11$ is 7. Show that the locus of the point is a st. line which makes equal $\angle$s with the given st. lines.

24. Find the distance between the $\parallel$ lines $\frac{x}{a} + \frac{y}{b} = c$, $\frac{x}{a} + \frac{y}{b} = d$.

25. Find the equation of the st. line passing through **P** $(2, 5)$ and cutting Ox at **A**, Oy at **B** so that **AP** : **PB** $= 7 : 3$.

26. Find the equations of the st. lines passing through $(4, 7)$ and making an $\angle$ of $45°$ with $3x - 10y = 8$.

27. Find the equations of the st lines passing through $(-4, -7)$ and forming an equilateral $\triangle$ with $3x - 2y = 7$.

28. Find the equations of the st. lines drawn $\parallel$ to $5x - 12y = 9$ and at a distance 5 from it.

29. Find the equations of the two st. lines which pass through $(4, 7)$ and are equally distant from **A** $(7, 3)$, **B** $(3, -1)$. Find also the distances from **A** and **B** to these lines.

30. Find the point in $4x - 3y = 12$ which is equally distant from $(2, 7)$ and $(4, -1)$.

31. Having given the length of the base and the difference of the squares of the other two sides of a △, prove that the locus of its vertex is a st. line ⊥ the base.

32. Find the equations of the st. lines which are at a distance 2 from the origin and which pass through the intersection of $x - 7y + 11 = 0$ and $3x + 4y - 17 = 0$.

33. Find the equation of the st. line ∥ to $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$ and at a distance p from the origin.

34. Find the equation of the st. line passing through (h, k) and ⊥ to $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$.

35. The equations of the sides of a △ are $5x + 3y - 15 = 0$, $2x - y + 4 = 0$, $3x - 7y - 21 = 0$. (*a*) Show that the ⊥s from the vertices to the opposite sides are concurrent and find the coordinates of the orthocentre. (*b*) Show that the right bisectors of the sides are concurrent and find the coordinates of the circumcentre. (*c*) Show that the centroid is at a point of trisection of the st. line joining the orthocentre to the circumcentre.

Exercises

(*b*)

1. What does the equation $2x^2 - 11xy + 12y^2 + 7x - 13y + 3 = 0$ become when the origin is changed to the point $(1, 1)$ the directions of the axes being unchanged? ✓

2. Transform the equation $x^2 + xy - 7x - 4y + 12 = 0$ to ∥ axes through $(4, -1)$. ✓

3. Find the point that must be taken as origin, the directions of the axes being unchanged, in order that the terms of the first degree in x and y may vanish from the equation $x^2 + y^2 + 5x - 9y + 17 = 0$. Find also what the equation becomes.

4. Show that the terms of the first degree in x and y will vanish from the expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, if the origin be changed to $\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)$, the directions of the axes being unchanged.

5. Transform the equation $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$ by rotating the axes through an ∠ of 30°.

6. Find what the equation $x^2 - y^2 = a^2$ becomes when the axes are turned through an ∠ of 45°, the origin remaining the same.

7. Show that the equation $x^2 + y^2 = a^2$ is not changed when the axes are turned through any ∠ a, the origin remaining the same.

8. Find what the equation $33x^2 - 34\sqrt{3}xy - y^2 = 0$ becomes when the axes are turned through an ∠ of 60°, the origin remaining the same.

9. Find the smallest positive ∠ through which the axes must be turned in order that the coefficient of xy in the equation $59x^2 + 24\,xy + 66y^2 = 250$ may vanish; and also find what the equation becomes.

10. Show that the term involving xy in the expression $ax^2 + 2\,hxy + by^2$ will vanish, if the axes are turned through the ∠.

$$\tfrac{1}{2}\, tan^{-1} \frac{2\,h}{a - b}.$$

58.—Review Exercises

1. Find the distances between the following pairs of points:— ✓

(*a*) $(-2,\ 7),\ (6,\ -2)$;

(*b*) $(2a + b,\ a - 2b),\ (a - b,\ 3a + b)$;

(*c*) $(a \cos a,\ a \sin a),\ (-b \cos a,\ -b \sin a)$.

Verify the result in (*b*), on squared paper, when $a = 1$, $b = -2$.

2. A $(-5,\ -1)$, B $(4,\ 6)$ are two given points, P is taken in AB and Q in AB produced such that AP : PB = AQ : QB = 5 : 3. Find the coordinates of P and Q.

3. Find the area of the △ of which the vertices are $(3a,\ 2b)$, $(2a,\ 3b)$ and $(a,\ b)$. ✓

4. Find the area of the △ contained by the lines $2x + 11y + 43 = 0$, $9x + 8y - 14 = 0$ and $7x - 3y + 26 = 0$.

5. Find the ⊥ distance from $(-2,\ 3)$ to the line ✓

$$\frac{2x}{3} - \frac{y}{2} = 6.$$

Should the result be considered positive or negative and why?

6. Find the condition that the three points $(x_1,\ y_1)$, $(x_2,\ y_2)$, $(x_3,\ y_3)$ may lie in a st. line.

7. Find the locus of a point such that the square of its distance from $(-3,\ -7)$ exceeds the square of its distance from $(5,\ 0)$ by 43.

8. Prove that the equation $\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0$ represents a st. line. ✓

9. Find the equation of the st. line which is equidistant from the ∥ lines $ax + by = c$, $ax + by = d$.

10. Find the equation of a st. line which makes an ∠ a with Oy and cuts off an intercept b from Ox.

11. Show that the st. lines $Ax + By + C = 0$, $A_1x + B_1y + C_1 = 0$ are ∥, if $AB_1 = A_1B$. ✓ $\because \frac{A}{B} = \frac{A_1}{B_1} \therefore m_1 = m$ ✓

12. Explain the meaning of the constants in the equations

$$\frac{x-h}{\cos\theta} = \frac{y-k}{\sin\theta} = r.$$

13. Show that the line $y = x \tan a$ passes through the point ($a \cos a$, $a \sin a$), and find the equation of the ⊥ to the line at that point.

14. Find the ∠ between the st. line joining $(-4, 5)$, $(5, 1)$ and the st. line joining $(3, 7)$, $(-6, -3)$.

15. Find the values of m and b such that the line $y = mx + b$ will pass through $(3, -2)$ and $(-1, -5)$.

16. Find the equation of the st. line which passes through $(2, -2)$, and makes an ∠ of 150° with Ox.

17. Find the length of the st. line drawn from (h, k), in the direction inclined at ∠ a to Ox, and terminated in the line $y = mx + b$.

18. Find the equation of the st. line through (h, k), and ∥ to $\frac{x}{a} + \frac{y}{b} = 1$. ✓

19. Show that the st. lines $Ax + By + C = 0$, $A_1x + B_1y + C_1 = 0$ are ⊥ to each other, if $AA_1 + BB_1 = 0$. ✓ $\frac{A}{B_1} = -\frac{A_1}{B}$

$= -1 \cdot \frac{A}{B}$

20. Write the equation of the st. line which is ⊥ $ax - by = c$, and cuts off an intercept $= d$ from Oy.

21. Find which of the following points are on the origin side of $\frac{x}{3} - \frac{y}{5} = 1$:— $(5, 3)$, $(-2, -8)$, $(2, -2)$, $(-6, -14)$, $(-7, -17)$. Illustrate by a diagram on squared paper. $\therefore \frac{A}{B_1} \cdot \frac{A_1}{B} =$

22. Show that the points $(2, 6)$, $(1, 11)$, $(-4, 7)$; $(-3, 3)$ are in the four different angular spaces made by the lines

$$\frac{x}{3} + \frac{y}{5} = 1 \text{ and } \frac{x}{5} - \frac{y}{9} = -1.$$

Illustrate by a diagram on squared paper.

23. Find the values of a for which the lines $2x - ay + 1 = 0$, $ax - 6y - 1 = 0$, $18x - ay - 7 = 0$ are concurrent; and find also the coordinates of the respective points of intersection.

24. Show that the condition that the lines $ax + by = 1$, $cx + dy = 1$, $hx + ky = 1$ are concurrent is the same as the condition that the the points (a, b), (c, d), (h, k) are collinear.

25. Find the $\angle$ contained by the lines $4x - 7y + a = 0$, $3x + 11y + b = 0$.

26. Find the equation of the st. line passing through the intersection of $4x - 7y + a = 0$ and $3x + 11y + b = 0$ and making an $\angle$ of 45° with the axis of x.

27. Find the equation of the st. line passing through the intersection of $\frac{x}{a} + \frac{y}{b} = 1$, $y = mx + c$ and also through $(d, 0)$.

28. Show that the equation of the st. line joining the intersection of $x \cos a + y \sin a = p$, $x \cos \beta + y \sin \beta = p$ to the origin is $y = x \tan \frac{a + \beta}{2}$.

29. Find the length of the $\perp$ from (a, b) to $\frac{x}{a} + \frac{y}{b} = 1$.

30. Find the equation of the st. line through $(-5, 1)$ and $\parallel$ to $3x + 12y = 17$.

31. Find the equation of the st. line through $(8, -2)$ and $\perp$ to $7x = y + 4$.

32. Find the coordinates of the four points each of which is equally distant from the three lines $\frac{x}{4} - \frac{y}{3} = 1$, $\frac{x}{12} + \frac{y}{5} = 1$, $\frac{x}{24} - \frac{y}{7} = -1$.

33. Find the coordinates of the foot of the ⊥ from $(3, 5)$ to the st. line joining $(-1, -2)$ and $(8, 1)$.

34. Show that the ∠ between the lines $y = mx + a$, $y = nx + b$ is $tan^{-1} \frac{m - n}{1 + mn}$.

35. Find the equations of the st. lines which pass through $(3, 6)$, and are inclined at an ∠ of 45° to $\frac{x}{5} + \frac{y}{7} = 1$.

36. Show that the st. line joining the point $(1, 1)$ to the intersection of $\frac{x}{a} + \frac{y}{b} = 1$ with $\frac{x}{b} + \frac{y}{a} = 1$ passes through the origin.

37. Show that the equation of the st. line passing through the intersection of $x \cos \alpha + y \sin \alpha = p$, $x \cos \beta + y \sin \beta = q$, and $\parallel$ to $x + y = k$ is

$$(x + y) \sin (\alpha - \beta) + p (\sin \beta - \cos \beta) + q (\cos \alpha - \sin \alpha) = 0.$$

38. Find the equation of the st. line passing through the intersection of $5x - 7y = 16$, $2x - 3y = 7$ and ⊥ to $6x - 4y = 19$. ✓

39. Find the equations of the st. lines drawn through the vertices and || to the opposite sides of the △ of which the equations of the sides are $3x + 11y = 23$, $4x - 9y = 11$, $7x - 2y = -31$.

40. Show that the lines $5x + y = 4$, $2x + y = 2$, $3x + 3y = 4$ are concurrent; and find the coordinates of their common point.

41. Find the equation of the st. line passing through the intersection of $ax + by + c = 0$, $fx + gy + h = 0$, and (a) also through the origin; (b) ⊥ to $x + y = k$.

42. Find the equations of the st. lines which bisect the ∠s between the lines $12x - 5y = 17$, $8x + 15y = 13$.

43. Find the equation of the st. line which passes through the point of intersection of the lines $5x + y = 4$, $4x - 9y = 11$, and is ⊥ to the former.

44. Show that the points (4, 2), (6, 2), $(5, 2 + \sqrt{3})$ are the vertices of an equilateral △.

45. Find the locus of a point which moves so that the sum of its distances from the axes is 10. Trace the locus on squared paper.

46. Find the locus of a point which moves so that the difference of its distances from the axes is 10. Trace the locus on squared paper.

47. Find the equations of the st. lines each of which passes through $(-5, -3)$ and is such that the part of it between the axes is divided at the given point in the ratio 7 : 3.

48. Find the equation of the st. line which passes through $(3, -2)$, and is ⊥ to $4x + y + 12 = 0$.

49. Find the equation of the right bisector of the st. line joining (a, b) and (h, k).

50. Two st. lines are drawn through $(0, -3)$ such that the $\perp$s on them from $(-6, -6)$ are each of length 3. Find the equation of the st. line joining the feet of the $\perp$s.

51. Find the $\angle$ of inclination of the lines $ax + by = c$, $(a + b)x - (a - b)y = d$.

52. A st. line is drawn through $(2, -4)$ and $\perp$ to $7x - 3y = 11$. Find the equations of the bisectors of the $\angle$s between the $\perp$ and the given st. line.

53. Three vertices of a $\parallel$gm are $(3, 4)$, $(-3, 1)$, $(5, -2)$. Find the coordinates of the fourth vertex.

54. Prove that the two st. lines which join the middle points of the opposite sides of any quadrilateral mutually bisect each other.

55. What must be the value of m, if the line $y = mx - 5$ passes through the intersection of $7x - 11y = 14$ and $5x + 2y = -11$.

56. Find the area of the $\triangle$ contained by the lines $x + y = 12$, $2x - y = 12$, $x - 2y = -12$.

CHAPTER III

THE CIRCLE

59. A **circle** is the locus of the points that lie at a fixed distance from a fixed point.

The fixed point is the **centre** and the fixed distance is the **radius** of the circle.

60. To find the equation of a circle having its centre at the origin.

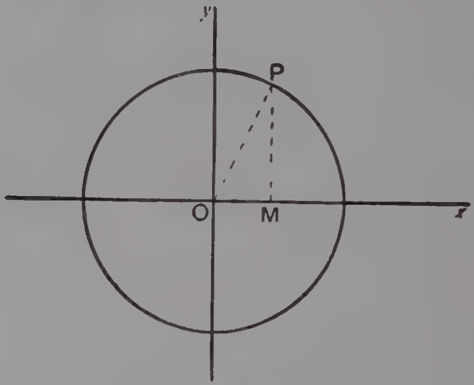


FIG. 32.

Let P (x, y) be any point on the circle of which the centre is O. Let the radius $= a$.

Draw PM $\perp$ Ox. Join PO.

$\because$ OPM is a rt.-$\angle$d $\triangle$,

$\therefore$ $OM^2 + PM^2 = OP^2$.

$\therefore$ $\mathbf{x^2 + y^2 = a^2}$.

This being the relation which holds between the coordinates of any point on the circle and the given radius is the required equation.

61. To find the equation of a circle, the centre being at any fixed point (h, k) and the radius equal to a.

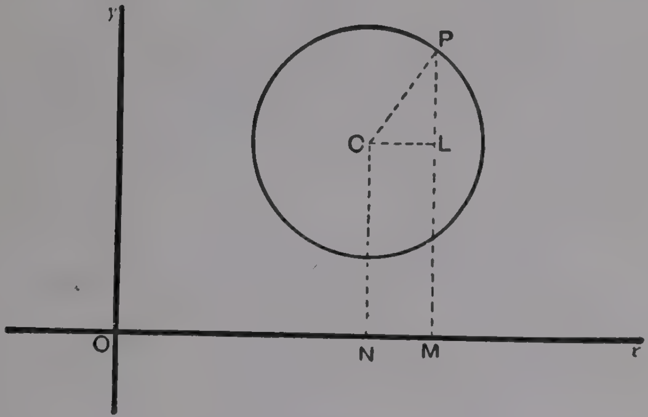


FIG. 33.

C (h, k) is the centre; and P (x, y) is any point on the circle.

Draw PM, CN $\perp$ Ox, CL $\perp$ PM. Join CP.

$$\text{CL} = \text{NM} = \text{OM} - \text{ON} = x - h;$$

$$\text{PL} = \text{PM} - \text{LM} = \text{PM} - \text{CN} = y - k.$$

$$\because \text{CPL is a rt.-}\angle\text{d } \triangle,$$

$$\text{CL}^2 + \text{PL}^2 = \text{CP}^2.$$

$$\therefore (x - h)^2 + (y - k)^2 = a^2.$$

This is the required equation.

62. If we expand the equation found in § 61, we obtain:—

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - a^2 = 0.$$

Comparing this result with the general equation of the second degree:—

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

we see that the conditions that the latter should represent a circle are that the coefficients of x^2 and y^2 should be equal and that the coefficient of xy should be zero.

Thus the equation

$$ax^2 + ay^2 + 2gx + 2fy + c = 0$$

may be changed to

$$\left(x + \frac{g}{a}\right)^2 + \left(y + \frac{f}{a}\right)^2 = \frac{g^2 + f^2 - ac}{a^2},$$

from which, by comparison with the formula of § 61, we see that it represents a circle having its centre at

m the point $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and its radius $= \frac{\sqrt{g^2 + f^2 - ac}}{a}$.

63. The general equation of the circle to rectangular axes is commonly written:—

$$\mathbf{x^2 + y^2 + 2gx + 2fy + c = 0}.$$

When the circle passes through the origin and its centre is on the axis of x, the equation of § 61 becomes

$$(x - a)^2 + y^2 = a^2,$$

or, $$\mathbf{x^2 + y^2 = 2ax}.$$

64.—Exercises

1. Write the equation of the circle with centre (0, 0) and radius $= \sqrt{5}$.

2. Write the equation of the circle with centre (6, 2) and radius $= 3$.

3. Write the equation of the circle with centre $(-5, -1)$ and radius $= \sqrt{26}$. Show that this circle passes through the origin.

4. Write the equation of the circle with centre $(-a, -b)$ and radius $= c$. Find the condition that this circle passes through the origin.

5. Find the coordinates of the centre and the radii of the following circles:—

(*a*), $x^2 + y^2 - 6x - 2y = 15$; (*b*), $4x^2 + 4y^2 + 7x + 5y = 16$; (*c*), $x^2 + y^2 = 14x$; (*d*), $x^2 + y^2 + 2\,by = c^2$.

6. Draw, on squared paper, the circles of which the equations are:—

(*a*), $x^2 + y^2 = 9$; (*b*), $x^2 + y^2 = 8x$; (*c*), $x^2 + y^2 + 6y = 7$.

7. Find the centre and radius of the circle which passes through the origin and cuts off intercepts $= a$ and b from **O**x and **O**y respectively.

Solution.—Since the circle passes through the origin its equation must be satisfied by $x = 0$, $y = 0$, and $\therefore$ the absolute term must be zero. Thus the equation may be written

$$x^2 + y^2 + 2\,gx + 2fy = 0.$$

Substituting in this equation the coordinates of the points $(a, 0)$, $(0, b)$ the two equations

$$a^2 + 2\,ga = 0$$
$$b^2 + 2\,fb = 0$$

are obtained from which $g = -\dfrac{a}{2}$, $f = -\dfrac{b}{2}$.

$\therefore$ the equation of the circle is

$$x^2 + y^2 - ax - by = 0,$$

$$\text{or, } \left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 = \frac{a^2 + b^2}{4}.$$

$\therefore$ the centre is $\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$ and the radius $= \dfrac{\sqrt{a^2 + b^2}}{2}$.

8. Find the equation of the circle which passes through the origin and also through $(4, 3)$ and $(-2, 6)$.

9. Find the equation of the circle which has its centre on the axis of x and which passes through the points $(5, 3)$ and $(-3, 1)$.

10. Show from the general equation of § 63 that three conditions are necessary and sufficient to determine a circle.

11. Find the condition that the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ may have its centre (*a*) on the axis of x; (*b*) on the axis of y.

12. Find the equation of the circle which passes through $(3, 1)$ and $(5, -3)$ and has its centre on the line $x - y = 4$.

13. Find the equation of the circle having the st. line joining $(7, -5)$ and $(-3, -1)$ as a diameter.

14. **A** $(a, 0)$ is a fixed point and **P** (x, y) is a variable point such that **PO** : **PA** $= p : q$. Show that the locus of **P** is a circle having its centre on **O**x, and dividing **OA** internally and externally in the ratio $p : q$.

15. Find the equation of the circumcircle of the △ whose vertices are $(3, 4)$, $(-2, 3)$, $(-5, -7)$.

16. Find the length of the chord of the circle $x^2 + y^2 = 25$ cut off by the line $3x + y = 15$.

17. Find the length of the chord of the circle $x^2 + y^2 - 6x + 14y = 42$ cut off by the line $x - y = 8$.

18. Show that the locus of the centres of all circles which pass through two given points (p, q), (r, s) is the right bisector of the st. line joining the given points.

19. Through the given point **P** (h, k) a st. line is drawn cutting the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at **A** and **B**. Prove that **PA . PB** is constant for all directions of the st. line.

Solution:—Take $\dfrac{x-h}{\cos\theta} = \dfrac{y-k}{\sin\theta} = r$ as the equation of the st. line. Then

$$x = h + r\cos\theta,\ y = k + r\sin\theta.$$

Substituting these values in the equation of the circle, and simplifying

$$r^2 + 2\left\{(h+g)\cos\theta + (k+f)\sin\theta\right\} r$$
$$+ h^2 + k^2 + 2gh + 2fk + c = 0.$$

The value of **PA . PB** = the product of the two values of r in this equation

$$= h^2 + k^2 + 2gh + 2fk + c,$$

an expression which does not contain θ and which is $\therefore$ independent of the direction of the line.

20. Find the equation of that chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which is bisected at the point (h, k).

Let the equation of the chord be

$$y - k = m(x - h).$$

The centre of the circle is $(-g, -f)$, and the equation of the $\perp$ from the centre to the chord is

$$m(y + f) + x + g = 0.$$

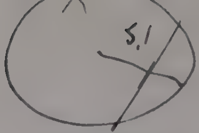
The $\perp$ from the centre bisects the chord, and, $\therefore$ passes through (h, k).

$$\therefore\ m(k + f) + h + g = 0.$$

$$\therefore\ m = -\frac{h+g}{k+f}.$$

$\therefore$ the required equation is

$$(h + g)(x - h) + (k + f)(y - k) = 0.$$



find slope of radius
thro 5, -1
∴ chord
subst.

21. Find the equation of the chord of the circle $x^2 + y^2 - 6x - 8y = 24$ which passes through $(5, -1)$ and is bisected at that point.

22. From the point **P** $(-3, -7)$ a st. line is drawn to cut the circle $x^2 + y^2 - 4x - 10y = 17$ at **A** and **B**. Find the area of the rectangle **PA . PB.**

23. Find the equation of the common chord of the circles

$$x^2 + y^2 - 8x - 6y = 39,$$
$$x^2 + y^2 + 6x + 8y = 56.$$

24. Find the condition that the common chord of the circles

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$
$$x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

passes through the origin.

25. Find the equation of the circle which passes through the origin and also through (h, k) and (k, h).

TANGENTS

65. Let APQ be a secant cutting a curve at P and Q.

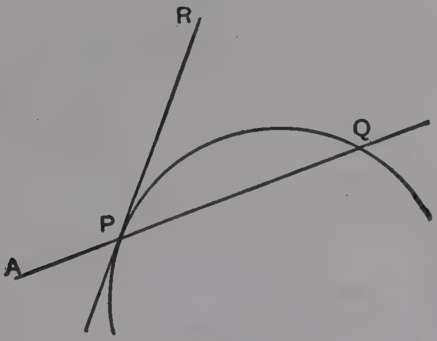


FIG. 34.

If the secant rotate about the point P until the second point Q approaches indefinitely near to P. the limiting position PR of the chord is called a **tangent** to the curve at the point P.

The point P is called the **point of contact** of the tangent PR.

66. To find the equation of the tangent to the circle $x^2 + y^2 = a^2$ at the point P (x_1, y_1) on the circle.

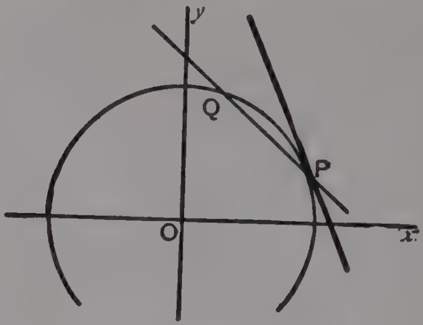


FIG. 35.

Let Q (x_2, y_2) be another point on the circle.

Then the equation of PQ is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}. \qquad (1)$$

∵ P and Q are both on the circle,

$$x_1^2 + y_1^2 = a^2,$$

$$\text{and, } x_2^2 + y_2^2 = a^2.$$

∴, subtracting, $x_1^2 - x_2^2 + y_1^2 - y_2^2 = 0.$

$$\therefore, (x_1 - x_2)(x_1 + x_2) + (y_1 - y_2)(y_1 + y_2) = 0 \qquad (2)$$

Multiplying the terms of (2) by the equal fractions in (1)

$$(x - x_1)(x_1 + x_2) + (y - y_1)(y_1 + y_2) = 0. \qquad (3)$$

If, now, PQ rotates about P until Q coincides with P, $x_2 = x_1$ and $y_2 = y_1$.

Thus equation (3) becomes

$$2(x - x_1)x_1 + 2(y - y_1)y_1 = 0,$$

$$\text{or,} \quad xx_1 + yy_1 = x_1^2 + y_1^2.$$

But $$x_1^2 + y_1^2 = a^2.$$

$$\therefore\ \mathbf{xx_1 + yy_1 = a^2}.$$ ✓

This is the required equation.

Alternative Method

The radius drawn from the point of contact of a tangent to the circle is perpendicular to the tangent.

The equation of the radius joining the origin to the point (x_1, y_1) on the circumference is

$$\frac{x}{x_1} = \frac{y}{y_1} \qquad \text{or } y_1x - x_1y = 0 \qquad (1)$$

Any line perpendicular to (*1*)

$$\text{is } xx_1 + yy_1 + K = 0$$

If it passes through (x_1, y_1)

$$\text{Then } x_1^2 + y_1^2 + K = 0$$

$$\text{or } a^2 + K = 0$$

$$\therefore\ K = -a^2$$

$\therefore$ The equation of the tangent becomes $xx_1 + yy_1 = a^2$

67. In any curve, the st. line drawn through the point of contact of a tangent and ⊥ to the tangent is called a **normal** to the curve at that point.

68. **To find the equation of the normal to the circle $x^2 + y^2 = a^2$ at the point x_1, y_1.**

The equation of the tangent at (x_1, y_1)

$$\text{is } xx_1 + yy_1 = a^2 \qquad (1)$$

Any line perpendicular to (*1*)

$$\text{is } y_1x - x_1y + K = 0$$

If it passes through (x_1, y_1)

$$\text{Then } x_1y_1 - x_1y_1 + \mathsf{K} = 0$$

$$\text{or } \mathsf{K} = 0$$

$\therefore$ the equation of the normal becomes

$$\left.\begin{aligned} y_1x - x_1y &= 0 \\ \text{or } \frac{x}{x_1} &= \frac{y}{y_1} \end{aligned}\right\} \checkmark$$

69. **To find the equation of the tangent to the circle $x^2 + y^2 = a^2$ in terms of m, the slope of the tangent.** ✓

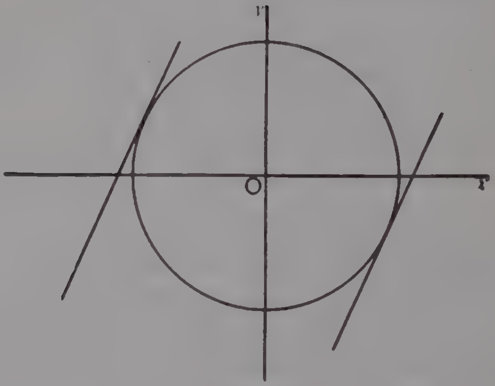


FIG. 36.

To find the abscissae of the points where the line $y = mx + k$ cuts the circle, eliminate y by substitution, and

$$x^2 + (mx + k)^2 = a^2,$$

$$i.e., \; (1 + m^2)\, x^2 + 2\, mkx + k^2 - a^2 = 0. \quad \checkmark$$

If the line is a tangent, the values of x from this equation are equal to each other, and

mx +k

solve for K

$$\therefore\ m^2k^2 = (1 + m^2)(k^2 - a^2),$$

$$\therefore\quad k^2 = a^2(1 + m^2),$$

$$\text{and } k = \pm a\sqrt{1 + m^2}.$$

Thus the equation of the tangent is

$$\mathbf{y = mx \pm a\sqrt{1 + m^2}}.$$

The double sign corresponds to the two tangents that have the same slope, as indicated in the diagram.

Alternative Method

Let equation of the tangent be $y = mx + k$

The perpendicular distance from the centre (o, o) on

$y = mx + k$ is $\dfrac{k}{\pm\sqrt{1 + m^2}}$

But the distance from the centre to the tangent $y = mx + k$ is a, the radius of the circle.

$$\therefore\ \frac{k}{\pm\sqrt{1 + m^2}} = a$$

or $k = \pm a\sqrt{1 + m^2}$

Substituting in $y = mx + k$ the equation of the tangent becomes $y = mx \pm a\sqrt{1 + m^2}$.

70.—Exercises

1. Find the equation of the tangent to the circle $x^2 + y^2 = 34$, at the point $(3, 5)$;

2. Find the equations of the st. lines touching the circle $x^2 + y^2 = 35$ and making an $\angle$ of $45°$ with the axis of x.

3. Find the equations of the st. lines which touch $x^2 + y^2 = r^2$ and are (*a*) || to, (*b*) ⊥ to the line $Ax + By + C = 0$

4. Prove that $x + 2y = 10$ is a tangent to the circle $x^2 + y^2 = 20$; and find the point of contact.

5. Find the condition that $\frac{x}{a} + \frac{y}{b} = 1$ may touch $x^2 + y^2 = r^2$.

6. Find the condition that the axis of x may touch $x^2 + y^2 + 2gx + 2fy + c = 0$.

7. Find the equations of the circles passing through (5, 2) and touching the axes of x and y.

8. Find the length of the part of the line $2x - 5y + 10 = 0$ intercepted by the circle $x^2 + y^2 + 8x - 6y = 24$.

9. Show that the tangent to the circle $(x - a)^2 + (y - b)^2 = r^2$ at the point (x_1, y_1) on this circle is $(x - a)(x_1 - a) + (y - b)(y_1 - b) = r^2$.

Solution.—Transform the origin to the point (a, b) without changing the direction of the axes. The transforming relations are $x = X + a$, $y = Y + b$, $x_1 = X_1 + a$, $y_1 = Y_1 + b$.

The equation of the circle becomes

$$X^2 + Y^2 = r^2,$$

and, by § 66 the tangent at (X_1, Y_1) is

$$XX_1 + YY_1 = r^2.$$

Transforming back to the original origin, the equation of the tangent becomes

$$(x - a)(x_1 - a) + (y - b)(y_1 - b) = r^2.$$

mem

10. Show that the circles

$$x^2 + y^2 + 6x + 16y + 24 = 0$$

$$x^2 + y^2 - 10x + 4y + 20 = 0$$

touch each other externally. i.e. have one pt. only in

11. Find the equation of the common chord; and the coordinates of the points of intersection of the circles

$$x^2 + y^2 - 2x - 6y = 14,$$

$$x^2 + y^2 - 5x + 3y = 5.$$

12. Find the equation of the circle whose centre is at the origin and which touches the line $\mathbf{A}x + \mathbf{B}y + \mathbf{C} = 0$.

13. Find the equation of the circle whose centre is at (7, 2) and which touches $3x - 5y = 4$.

14. Find the equations of the two tangents to the circle $x^2 + y^2 = 25$ which make an ∠ of 30° with the axis of x.

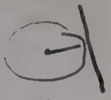
15. Find the equation of the tangent to the circle

cf. 9

(*a*) $x^2 + y^2 - 10x + 12y = 39$ at $(-1, 2)$;

(*b*) $x^2 + y^2 + 18x - 14y = 39$ at $(3, 12)$;

(*c*) $x^2 + y^2 + 2gx + 2fy = 0$ at the origin.



find eq. of r

16. Prove that $x - 2y = 4$ is a tangent to the circle $x^2 + y^2 - 8x - 10y + 21 = 0$.

f. 5

17. Find the condition that $\mathbf{A}x + \mathbf{B}y + \mathbf{C} = 0$ may touch $x^2 + y^2 = r^2$.

POLES AND POLARS

71. To find the equation of the chord of contact of tangents drawn from an outside point to the circle $x^2 + y^2 = a^2$.

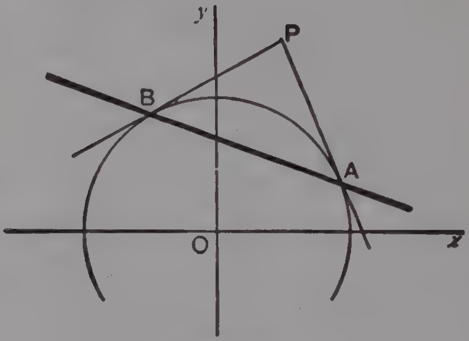


FIG. 37.

Let **P** (x_1, y_1) be the given point; **PA** and **PB** the tangents.

It is required to find the equation of **AB**.

Take (x', y'), (x'', y'') to represent the coordinates of **A**, **B** respectively.

The equation of **AP** is, by § 66,

$$xx' + yy' = a^2;$$

and since the coordinates of **P** must satisfy this equation,

$$x_1x' + y_1y^1 = a^2. \qquad (1)$$

Similarly, $$x_1x'' + y_1y'' = a^2. \qquad (2)$$

From these results it is seen that

$$xx_1 + yy_1 = a^2$$

is the equation of AB; for:—

since it is of the first degree it represents a st. line;

by (1), A is a point on the line;

by (2), B is a point on the line;

∴ the required equation is

$$\mathbf{xx_1 + yy_1 = a^2.}$$

72. The equation of the chord of contact of tangents drawn from an outside point to a circle is of the same form as the equation of the tangent at a point on the circle.

This is in agreement with the fact that, if the point P approach the circle and ultimately fall on it, the chord of contact becomes the tangent at P, or the tangent at P is the final position of the chord of contact when P approaches the circle.

73. Definition.—The **polar** of any point P with respect to a circle is the locus of the intersection of tangents drawn at the ends of any chord which passes through P. The point P is called the **pole** of the locus.

74. To find the equation of the polar of P (x_1, y_1) with respect to the circle $x^2 + y^2 = a^2$.

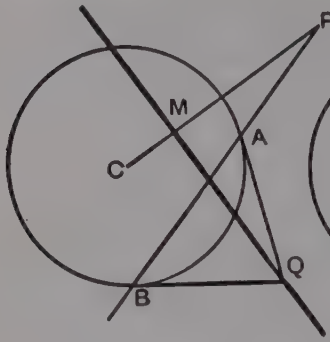


FIG. 38.

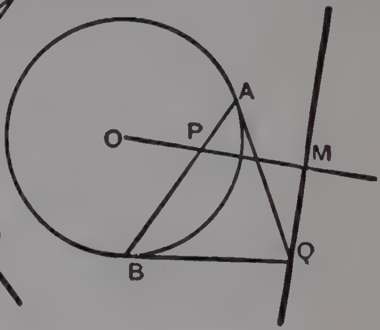


FIG. 39.

Through P draw any st. line cutting the circle at A, B. Draw tangents AQ, BQ intersecting at Q (X, Y).

It is required to find the locus of Q.

By § 71, the equation of AB is

$$x\mathrm{X} + y\mathrm{Y} = a^2;$$

and as the coordinates of P must satisfy this equation

$$\mathrm{X}x_1 + \mathrm{Y}y_1 = a^2.$$

∴, as X, Y are the coordinates of any point on the polar of P, the required equation is

$$\mathbf{x}\mathbf{x}_1 + \mathbf{y}\mathbf{y}_1 = \mathbf{a}^2.$$

75. The equation of OP is $xy_1 - yx_1 = 0$, and, by the condition for perpendicularity, this line is ⊥ to that represented by $xx_1 + yy_1 = a^2$.

∴ **the polar of P is a st. line which cuts OP at rt. ∠s**

76. If the polar $xx_1 + yy_1 = a^2$ cuts **OP** at **M**, the length of **OM**

$$= \frac{a^2}{\sqrt{x_1^2 + y_1^2}} = \frac{a^2}{\text{OP}}.$$

$$\therefore \ \text{OM} \,.\, \text{OP} = a^2.$$

77. The equation of the polar of the point **P** (x_1, y_1) without the circle $x^2 + y^2 = a^2$ is the same as that of the chord of contact of tangents drawn from **P** to the circle. This shows that, when the point is without the circle, its polar is the chord of contact produced, or, that **tangents drawn from P touch the circle at the points where it is cut by the polar of P.**

78. The equation of the polar of any point **P** (x_1, y_1) is of the same form as the equation of the tangent at a point on the circle.

This is in agreement with the fact that, if the point **P** approaches and ultimately coincides with the circle, **OP** becomes equal to a, and $\therefore$, by § 76, **OM** becomes equal to a, **and the polar becomes the tangent at the point P**

79. If the polar of P passes through Q, the polar of Q passes through P.

Let (x_1, y_1), (x_2, y_2) be the coordinates of P, Q respectively.

The polar of P with respect to $x^2 + y^2 = a^2$ is

$$xx_1 + yy_1 = a^2.$$

Since this line passes through Q,

$$x_2x_1 + y_2y_1 = a^2.$$

This proves that (x_1, y_1) is on the line

$$xx_2 + yy_2 = a^2;$$

and $\therefore$ P is on the polar of Q.

Cor. If the point Q moves along the polar of P, the polar of Q changes its position, but always passes through P.

$\therefore$, if the pole moves along a st. line, its polar turns about the pole of that line.

80. To find the pole of the st. line $Ax + By + C = 0$ with respect to the circle $x^2 + y^2 = a^2$.

Let (x_1, y_1) be the coordinates of the pole

The equation of the polar of (x_1, y_1) is

$$xx_1 + yy_1 - a^2 = 0.$$

This equation must be the same as

$$Ax + By + C = 0.$$

$$\therefore \quad \frac{x_1}{A} = \frac{y_1}{B} = \frac{-a^2}{C}$$

$$\therefore \quad x_1 = -\frac{a^2A}{C}, \; y_1 = -\frac{a^2B}{C}.$$

81.—Exercises

1. Find the polar of the point

(*a*) $(3, 5)$ with respect to $x^2 + y^2 = 30$;
(*b*) $(a, 0)$ " " " $x^2 + y^2 = r^2$;
(*c*) $(-2, 4)$ " " " $x^2 + y^2 - 4x - 8y = 5$;
(*d*) $(-5, -1)$ " " " $x^2 + y^2 - 10x + 6y = 15$;
(*e*) $(0, 0)$ " " " $(x - h)^2 + (y - k)^2 = r^2$.

2. Find the pole of the st. line

(*a*) $2x - 7y = 17$ with respect to $x^2 + y^2 = 17$;
(*b*) $x - 2y + 12 = 0$ " " " $x^2 + y^2 = 23$;
(*c*) $4x - y = 1$ with respect to $x^2 + y^2 - 2x - 4y = 4$;
(*d*) $4x + 5y = 5$ " " " $x^2 + y^2 - 8x - 10y = -5$.

3. (*a*) Show that $x^2 + y^2 = 25$ is the equation of a circle.

(*b*) Show that $(-3, 4)$ is on the circle.

(*c*) Write the equation of the tangent to the circle at this point.

(*d*) Show that the point $(9, 13)$ is on this tangent.

(*e*) Write the equation of the polar of $(9, 13)$.

(*f*) Find the equation of the st. line through $(9, 13)$ $\perp$ to the polar, commenting on the form of the result.

(*g*) Find the equation of the other tangent from $(9, 13)$.

Draw the diagram on squared paper.

4. Find the pole of $\dfrac{x}{a} + \dfrac{y}{b} = 1$ with respect to $x^2 + y^2 = c^2$.

TANGENTS FROM AN OUTSIDE POINT

82. To find the length of the tangent PA from the point P (x_1, y_1) to a given circle.

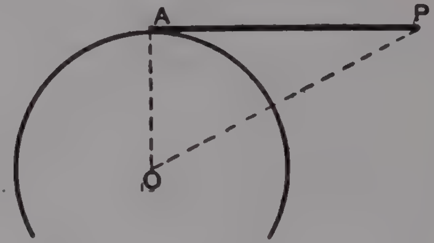


FIG. 40.

(1) Let the equation of the circle be

$$x^2 + y^2 = a^2.$$

Join OA, OP.

$$\because \quad AOP \text{ is a rt.-}\angle\text{d } \triangle,$$

$$AP^2 = OP^2 - AO^2$$

$$= x_1^2 + y_1^2 - a^2.$$

$$\therefore \quad AP = \sqrt{x_1^2 + y_1^2 - a^2}.$$

(2) Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

This equation may be written

$$(x + g)^2 + (y + f)^2 = g^2 + f^2 - c,$$

from which it is seen that the centre is $(-g, -f)$ and the radius $= \sqrt{g^2 + f^2 - c}$.

With the diagram and construction of Fig. 40,

$$AP^2 = OP^2 - AO^2$$

$$= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$$

$$= x_1^2 + y_1^2 + 2\,gx_1 + 2fy_1 + c.$$

$$\therefore \quad AP = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}.$$

83.—Exercises

1. Find the length of the tangent from

(*a*), $(7, 3)$ to $x^2 + y^2 = 22$;

(*b*), $(3, -5)$ to $x^2 + y^2 - 3x + 7y + 35 = 0$;

(*c*), $(-2, -6)$ to $x^2 + y^2 = 12x$;

(*d*), $(0, 0)$ to $x^2 + y^2 + 2gx + 2fy + c = 0$;

(*e*), $(4, 2)$ to $x^2 + y^2 - 6x + 2y - 39 = 0$.

Explain the imaginary result in (*e*).

2. The length of the tangent drawn from a point to $x^2 + y^2 - 10x - 4y + 9 = 0$ is always 4. Find the locus of the point. Plot the diagram on squared paper.

3. The length of the tangent from P to $x^2 + y^2 = 9$ is twice the distance from P to $(6, 0)$. Find the locus of P.

4. Find the equations of the tangents from $(7, -1)$ to $x^2 + y^2 = 25$.

Miscellaneous Exercises

1. Find the equation of the st. line passing through the intersection of $x - 2y = 5$, $x + 3y = 10$ and $\parallel$ to $3x + 4y = 11$.

2. Find the equation of the st. line passing through the intersection of $8x + y = 7$, $11x + 2y = 28$ and $\perp$ to the latter line.

3. Plot the quadrilateral $(4, 2)$, $(-5, 6)$, $(-9, -6)$, $(7, -4)$; and find its area.

4. Plot the lines $2x + 5y = 29$, $12x + y = 29$, $5x - 2y = 29$; and find the area contained by them.

5. Find the equation of the st. line passing through (h, k) and such that the portion of it between the axes is bisected at the given point.

6. Find the equation of the st. line passing through (h, k) and (*a*) ∥ to, (*b*) ⊥ to the st. line joining (x_1, y_1), (x_2, y_2).

7. Show that, if the lines $ax + by + c = 0$, $bx + cy + a = 0$, $cx + ay + b = 0$ are concurrent but not coincident, then $a + b + c = 0$.

8. Find the ratio in which the st. line joining $(-5, 3)$, $(6, -1)$ is divided by $x - 11y + 3 = 0$.

9. Find the centre of the inscribed circle of the △ formed by the lines $4x - 3y = 18$, $5x + 12y = 9$, $24x + 7y = 30$.

10. Find the area of the △ contained by $y = 3x$, $y = 5x$ and $x + 2y = 77$.

11 Show that the area of the △ contained by $y = m_1x$, $y = m_2x$ and $Ax + By + C = 0$

$$= \frac{(m_1 - m_2)\, C^2}{2\,(A + m_1B)\,(A + m_2B)}.$$

12. Find the locus of a point such that the square of its distance from $(6, 0)$ is three times the square of its distance from $(2, 0)$.

13. One vertex of a ∥gm is at the origin and the two adjacent vertices are at (a, b), (c, d). Find the fourth vertex.

14. Show that, if the two circles

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$
$$x^2 + y^2 - 2fx - 2gy + c = 0$$

touch each other, then $(g - f)^2 = 2c$.

15. Find the locus of the intersection of the st. lines which pass through $(6, 0)$ and $(0, 3)$ respectively and cut each other at rt. ∠s.

16. Find the equations of the tangents to the circle $x^2 + y^2 = 2kx$ which are || to $3x - y = 0$.

17. Find the orthocentre of the △ whose sides are $8x - 5y = 16$, $2x - 3y = 10$ and $x + 2y = 6$.

18. Chords of the circle $x^2 + y^2 = a^2$ pass through the fixed point (h, o). Find the locus of their middle points.

19. Find the equation of the circle which passes through the origin and makes intercepts a and b on Ox, Oy respectively.

20. Find the equation of the circle described on the st. line joining the origin to (g, f) as diameter.

21. Find the coordinates of a point such that the st. line joining it to $(4, -3)$ is bisected at rt. ∠s by $2x - 3y = 7$.

22. Find the locus of the points from which tangents drawn to $x^2 + y^2 = 13$ and $x^2 + y^2 - 2x + 6y + 1 = 0$ are as 5 is to 3.

23. Find the distances from the point $(2, 4)$ in the direction having the direction cosines $-\frac{3}{5}, -\frac{4}{5}$ to the curve whose equation is

$$3x^2 - 6xy + 5y^2 - 32 = 0.$$

24. (*a*) Find the coordinates of the point C which divides the st. line joining A $(3, -2)$, B $(19, 10)$ in the ratio AC : CB = 1 : 3.

(*b*) Prove that D $(11, 4)$ lies on the st. line AB given above; and by computing the lengths of AD and BD, find the ratio in which D divides AB.

25. Show that $y = mx + a\sqrt{1 + m^2}$ is always a tangent to the circle $x^2 + y^2 = a^2$.

26. Find the distance of the point of intersection of the lines $3x + 2y + 4 = 0$ and $2x + 5y + 8 = 0$ from the line $5x - 12y + 6 = 0$.

27. Find the equation of the circle whose centre is (h, k) and which passes through (a, b).

28. Find the equation of the st. line joining $(ab^2, 2ab)$ and $(ac^2, 2ac)$.

29. Show that the equation of the $\perp$ to $\frac{\cos a}{a}x + \frac{\sin a}{b}y = 1$ at the point $(a \cos a, b \sin a)$ is $\frac{a}{\cos a}x - \frac{b}{\sin a}y = a^2 - b^2$.

CHAPTER IV

THE PARABOLA

84. **A Conic Section** or **Conic** is the locus of a point which moves so that its distance from a fixed point called the **focus** is in a constant ratio to its distance from a fixed straight line called the **directrix.** The constant ratio is called the **eccentricity** and is usually denoted by e.

If the eccentricity, e, is equal to unity the conic is a parabola.

If the eccentricity, e, is less than unity the conic is an ellipse.

If the eccentricity, e, is greater than unity the conic is an hyperbola.

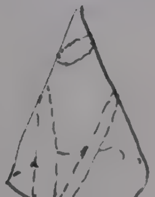
The point midway between the focus and directrix is called the **vertex.**

Historically the parabola, ellipse and hyperbola were discovered as plane sections of a right circular cone. If a right circular cone be cut by any plane the curve of section will in all cases be a conic as defined above.

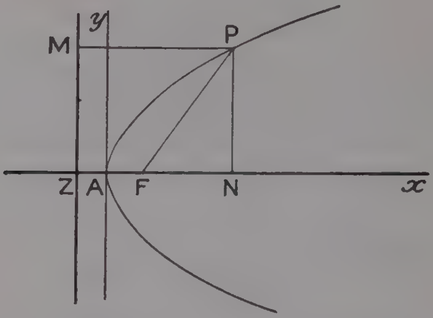
Use a cone of modelli wax (an Knife)

The parabola ($e = 1$) is the simplest of these curves; the line with respect to which this curve is symmetrical is called its **axis.**

85. **To find the equation of the Parabola having its vertex at the origin and its axis coinciding with the x-axis.**



As $e = 1$ in the parabola the distance of any point on the curve from the focus is equal to its distance from the directrix.



Let P (x, y) be any point on the curve; F the focus; ZM the directrix.

Draw ZF perpendicular to ZM and let ZF $= 2a$.

Bisect ZF at A. Then ZA = AF $= a$.

Take A as origin, Ax as axis of x, and Ay as axis of y.

Draw PN, PM perpendicular to Ax, MZ respectively.

By definition $\quad \text{FP} = \text{MP} = \text{ZN}.$

$$\therefore \text{FP}^2 = \text{ZN}^2.$$

$$\therefore \text{FN}^2 + \text{NP}^2 = (\text{ZA} + \text{AN})^2.$$

$$\text{or } (x - a)^2 + y^2 = (a + x)^2.$$

Whence $y^2 = 4ax$, which is the required equation.

The point A is called the Vertex;

The co-ordinates of the focus are a, o.

The equation of the directrix is $x = -a$ or $x + a = o$.

Similarly the equation of the parabola whose vertex is at the origin and whose axis coincides with the y-axis is $x^2 = 4ay$.

86. In deducing the equation of the parabola $y^2 = 4ax$, the focus and directrix were placed in convenient positions in order to get the equation in the most suitable form from which to study the properties of the parabola. In the following examples the focus is taken as any point and the directrix as any straight line.

(*1*) **To find the equation of a parabola whose focus is $(1, 2)$ and directrix is $3x + 4y + 5 = 0$.**

Let (x, y) be any point on the curve.

Then by the definition of a parabola the distance of (x, y) from the focus equals its distance from the directrix; the equation of the parabola is therefore

$$\sqrt{(x-1)^2 + (y-2)^2} = \frac{3x + 4y + 5}{\pm 5}$$

$$\text{or } 16x^2 - 24xy + 9y^2 - 80x - 140y + 100 = 0.$$

(*2*) **Taking any point (h, k) as the focus and any straight line $Ax + By + C = 0$ as the directrix.**

The equation of the parabola is

$$(x-h)^2 + (y-k)^2 = \frac{(Ax + By + C)^2}{A^2 + B^2}.$$

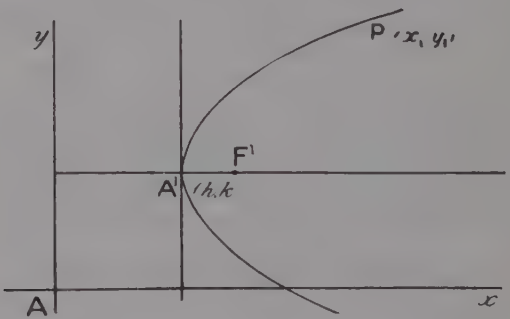
87. The equation $y^2 = 4ax$, of the parabola is the one generally used. In deducing this equation we supposed that the positions of the focus and directrix were given. The distance, $(ZF = 2a)$, from the focus to the directrix is the quantity that distinguishes this parabola from other parabolas.

In the parabola $y^2 = 4ax$, a is a known quantity just as in the circle $x^2 + y^2 = r^2$ the radius r is a known quantity.

Such quantities as r and a are called parameters.

omit

88. To find the equation of the parabola whose vertex is at (h, k) and whose axis is parallel to the X axis.



Let P (x_1, y_1) be any point on the curve, x_1, y_1 being the co-ordinates with respect to the new axis.

Then (by § 54)

$$x = x_1 + h$$

$$y = y_1 + k$$

$$\text{or} \quad x_1 = x - h$$

$$\text{and} \quad y_1 = y - k.$$

Substituting in $y_1^2 = 4ax_1$

$$(y - k)^2 = 4a(x - h).$$

Similarly the equation of the parabola whose vertex is at the point (h, k) and whose axis is parallel to the y-axis is

$$(x - h)^2 = 4a(y - k).$$

From the above equations it is evident that:—

(*1*) the equation of any parabola whose axis is parallel to the x-axis is of the form,

$$\mathsf{B}y^2 + \mathsf{D}x + \mathsf{E}y + \mathsf{C} = 0.$$

(*2*) the equation of any parabola whose axis is parallel to the y-axis is of the form,

$$\mathsf{A}x^2 + \mathsf{D}x + \mathsf{E}y + \mathsf{C} = 0.$$

89. If the equation of a parabola is given, it is possible to find its vertex, the equation of its axis, the co-ordinates of its focus and the equation of its directrix.

Example:

Let $y^2 + y - 3x + 1 = 0$ be the equation of a parabola.

$$\text{If } y^2 + y - 3x + 1 = 0.$$

$$\text{then } y^2 + y + \tfrac{1}{4} = 3x - 1 + \tfrac{1}{4}.$$

$$\text{or } (y + \tfrac{1}{2})^2 = 3(x - \tfrac{1}{4})$$

Expressing this equation in the form

$$(y - k)^2 = 4a(x - h)$$

$$\text{it becomes } [y - (-\tfrac{1}{2})]^2 = 4.\tfrac{3}{4}(x - \tfrac{1}{4}).$$

Hence (*1*) the co-ordinates of the vertex are $\frac{1}{4}, -\frac{1}{2}$,

(2) the equation of the axis of the parabola is $y = -\frac{1}{2}$ or $y + \frac{1}{2} = 0$ or $2y + 1 = 0$.

From the figure in § 88 it is evident that the co-ordinates of the focus F^1 are $(h + a, k)$.

(*3*) The co-ordinates of the focus of this parabola are

$$\left(\tfrac{1}{4} + \tfrac{3}{4}, -\tfrac{1}{2}\right) \text{ or } \left(1, -\tfrac{1}{2}\right).$$

(*4*) To find the equation of the directrix,

Changing the equation $[y - (-\frac{1}{2})]^2 = 4.\frac{3}{4}(x - \frac{1}{4})$ so that the vertex is at the origin it becomes $y^2 = 4.\frac{3}{4}x$.

Evidently the equation of the directrix with vertex at the origin is $x = -\frac{3}{4}$.

Changing this line to the original axes the equation of directrix becomes $x = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$

$$\text{or } 2x + 1 = 0.$$

The above method for finding the equation of the directrix is usually shortened thus,

$$x - \tfrac{1}{4} = -\tfrac{3}{4}$$

$$x = -\tfrac{1}{2} \qquad (\S\ 88)$$

$$\text{or } 2x + 1 = 0.$$

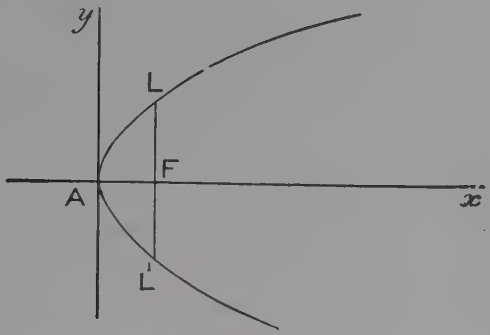
90. To find the distance of any point on the parabola, $y^2 = 4ax$, from the focus.

Using the figure of § 85.

$$\begin{aligned}\text{The distance of } P \text{ from the focus} &= PF\\ &= MP\\ &= ZN\\ &= a + x.\end{aligned}$$

91. The double ordinate through the focus of a conic is called the **latus rectum.**

To find the length of the latus rectum of the parabola.



LFL^1 is the latus rectum of the parabola.

The co-ordinates of L are (a, FL) and $AF = x = a$

Substituting in $y^2 = 4ax$

$$FL^2 = 4a.\ a = 4a^2$$

$$FL = 2a$$

$\therefore$ latus rectum $LFL^1 = 4a.$

92.—Exercises

1. Plot the locus of the following equations:

(a) $y^2 = 4x.$
(b) $y^2 + 4x = 0.$
(c) $x^2 = 8y.$
(d) $x^2 + 8y = 0.$
(e) $y^2 + 4x + 4y - 2 = 0.$
(f) $x^2 + 6x + y - 2 = 0.$

2. Find the position of the focus and length of the latus rectum in the following parabolas:

$$(a)\ y^2 + 4ax = 0.$$
$$(b)\ x^2 = -4ay.$$
$$(c)\ y^2 = \tfrac{1}{16}x.$$

3. Find the equation of the parabola taking the directrix as the axis of y and the axis of the curve as the axis of x.

4. Find the equation of the parabola taking the axis of the curve as the axis of y and the tangent at the vertex as the axis of x.

5. Find the equation of the parabola referred to its axis and latus rectum as axes of co-ordinates.

6. If the distance of a point on the parabola $y^2 = 4ax$ from its focus, be $6a$, find its co-ordinates.

7. Find the points where the line $y = 3x - a$ cuts the parabola $y^2 = 4ax$.

8. Find where the line $12x - 7y + 12 = 0$ cuts the parabola $y^2 = 12x$.

9. Determine a so that the parabola $y^2 = 4ax$ shall pass through the point $(-2, 4)$.

10. The three vertices of an equilateral triangle are on the parabola, $y^2 = 4ax$, one of them being the vertex of the parabola. Find the length of its side.

11. Find the length of the chord which the parabola, $y^2 = 8x$, intercepts on the line $2x + y = 8$.

12. A circle passes through the origin and the extremities of the latus rectum of the parabola, $y^2 = 4ax$. Find the equation of the circle.

13. Find the equations of the following parabolas:

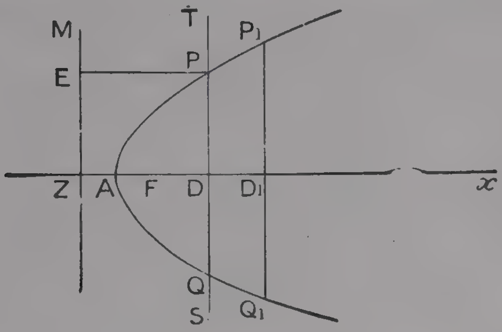
(a) focus $(2, 3)$ directrix $x + y = 2$.
(b) focus $(3, 5)$ directrix $24x + 7y + 18 = 0$.
(c) focus $(3, -3)$ directrix $5x + 12y = 13$.
(d) focus (c, d) directrix $kx + ly + m = 0$.

14. Find the focus and directrix of each of the parabolas:

$$x^2 + y^2 = \frac{(3x + 4y + 5)^2}{25}.$$

$$(x - 2)^2 + (y - 3)^2 = \frac{(5x + 12y + 7)^2}{169}.$$

93. Mechanical method for constructing a parabola.



If the focus and directrix of a parabola are given, its locus may be readily constructed by means of a ruler and compass.

Let **MZ** be the directrix and **F** the focus.

Through **F** draw **Z**x perpendicular to **MZ**.

Bisect **ZF** at **A**. Then **A** is the vertex of the parabola.

Through any point **D** on **Z**x to the right of **A** draw **TS** parallel to **MZ**.

With centre **F** and radius **ZD** describe arcs meeting **TS** in **P** and **Q**.

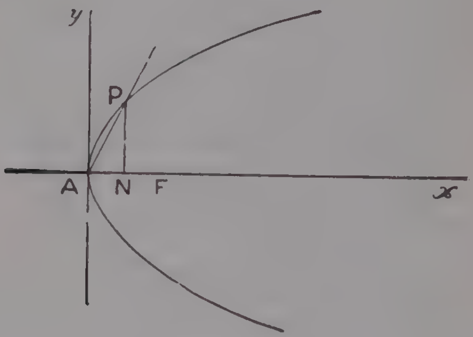
Then **P** and **Q** are points on the parabola. For **FP** = **ZD** = **EP**.

Hence the points **P** and **Q** are equidistant from the focus and directrix.

Letting **D** take any position on **A**x, pairs of points P_1, Q_1; P_2, Q_2; etc. equidistant from focus and directrix can be found.

The curve through these points is the parabola.

94. To trace the form of the parabola $y^2 = 4ax$ from its equation.



(*1*) If $x = 0$ then $y = 0$; hence the curve passes through the origin.

(*2*) Since y^2 is always positive and a being supposed positive it follows that x is always positive. Hence the curve does not exist to the left of the origin.

(*3*) As $y^2 = 4ax$ then $y = \pm 2\sqrt{ax}$.

For a given value of x the values of y are equal in magnitude, one being positive, the other negative. Hence the curve is symmetrical with respect to the line **A**x. This line **A**x is called its axis. Also all chords perpendicular to the axis of a parabola are bisected by it.

(*4*) As x increases, y increases; when x becomes indefinitely great, y is indefinitely great. Hence the curve consists of infinite branches above and below its axis.

(*5*) If the parabola $y^2 = 4ax$ be cut by the line $y = mx + b$ the abscissae of the common points are given by the equation $m^2 x^2 + 2(bm - 2a)x + b^2 = 0$. Since this is a quadratic equation, x has two values. Therefore a straight line can cut a parabola in only two points. These points may be real, coincident or imaginary.

From the theory of quadratics the product of the roots of the equation $m^2 x^2 + 2(bm - 2a)\ x + b^2 = 0$ is $\frac{b^2}{m^2}$.

Hence if m is zero one root is infinitely great.

But if $m = 0$ the line, $y = mx + b$, is parallel to the x-axis. Hence every straight line parallel to the axis of a parabola meets the curve in one point at a finite distance and in another at an infinite distance from the vertex.

(*6*) If P (x, y) be any point on curve, § 94,

$$\tan \text{PAN} = \frac{\text{NP}}{\text{AN}} = \frac{y}{x} = \frac{y^2}{xy} = \frac{4ax}{xy} = \frac{4a}{y}.$$

Ultimately when P moves along the curve indefinitely close to A the value of $y = 0$.

Then $\tan \text{PAN} = \frac{4a}{0} = \infty$.

Therefore angle PAN in the limit $= 90°$.

Hence the curve cuts the axis of x at right angles.

95.—Exercises

1. Find the co-ordinates of the vertex, the co-ordinates of the focus, the equation of the axis and the equation of the directrix in each of the following parabolas:

(*a*) $y^2 = 8x$.
(*b*) $(y - 1)^2 = 4(x - 2)$.
(*c*) $(y - 3)^2 = 8x + 11$.
(*d*) $x^2 + 4y + 8 = 0$.
(*e*) $x^2 + 2ax + 4by + a^2 + 4bc = 0$.
(*f*) $y^2 + 4y - 6x + 7 = 0$.
(*g*) $x^2 + 6x + y - 2 = 0$.
(*h*) $4y^2 + 4y + 3x - 2 = 0$.
(*i*) $2x^2 - 6x + y = 0$.

2. The vertex of a parabola is at the point (2, 3) and the parabola passes through the origin. Find its equation, its axis being parallel to the x-axis.

3. The vertex of a parabola is at the point $(-1\frac{1}{2}, 2)$ and the parabola passes through the point $(-1, -1)$. Find its equation, its axis being parallel to the y-axis.

4. The vertex of a parabola is (3, 2) and its focus is (5, 2). Find its equation.

5. Find the equation of the parabola when the focus is at the origin and the axis coincides with the x-axis.

6. Find the equation of the parabola when the axis and directrix coincide respectively with the x-axis and y-axis.

7. Find the equation of the parabola whose focus is $(-1, 3)$ and directrix the line $x - 1 = 0$.

8. Find the equation of the parabola, the vertex being $(-2, -5)$ and the directrix the line $x - 3 = 0$.

9. Find the equation of the parabola whose vertex is $(1, -2)$, its axis being parallel to the x-axis, and the distance from the vertex to the focus equals 2.

10. Find the equation of the parabola passing through (2, 1) whose vertex is (0, 2), and whose directrix is parallel to the x-axis.

11. Find the equation and length of the latus rectum of the parabola whose focus is $(-a, o)$ and directrix is $x + y + 2a = 0$.

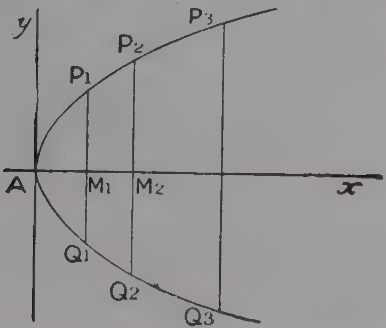
12. Find the equation of the circle through the vertex, the focus and the upper end of the latus rectum of the parabola $y^2 = 8x$.

13. Plot the points (o, o), $(1, 0.5)$, $(2, 2)$, $(3, 4.5)$ $(4, 8)$, $(5, 12.5)$, taking the half inch as the unit.

Prove that a parabola with the y-axis for axis of the curve can be drawn through these points. Find its equation and draw it.

14. Prove that the product of the segments of any focal chord of the parabola $y^2 = 4ax$ equals "a" times the length of the chord.

96. The squares of any two chords of a parabola which are perpendicular to the axis of the parabola are to each other as their distances from the vertex.



Let $P_1 (x_1, y_1)$ and $P_2 (x_2, y_2)$ be any two points on the parabola $y^2 = 4ax$, and P_1Q_1 and P_2Q_2 be any two chords perpendicular to the axis of the parabola.

Then
$$y_1^2 = 4ax_1.$$
$$y_2^2 = 4ax_2.$$
$$\therefore \frac{y_1^2}{y_2^2} = \frac{x_1}{x_2}.$$

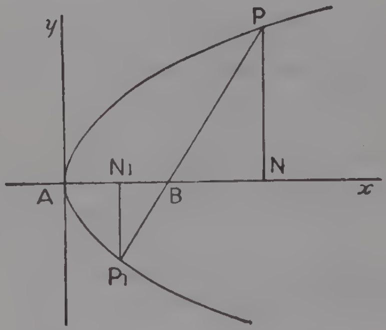
Hence
$$\frac{(2y_1)^2}{(2y_2)^2} = \frac{x_1}{x_2}. \qquad (1)$$

From the symmetry of the parabola, $2y_1 = \mathsf{P_1 Q_1}$ and $2y_2 = \mathsf{P_2 Q_2}$; also $x_1 = \mathsf{AM_1}$ and $x_2 = \mathsf{AM_2}$.

Hence (*1*) becomes

$$\frac{\mathsf{P_1\,Q_1}^2}{\mathsf{P_2\,Q_2}^2} = \frac{\mathsf{AM_1}}{\mathsf{AM_2}}.$$

97. **If a chord passes through a fixed point on the axis of a parabola, the product of the abscissae of its extremities is constant.**



Let $\mathsf{P}\,(x, y)$ and $\mathsf{P_1}\,(x_1, y_1)$ be the extremities of a chord through the fixed point $\mathsf{B}\,(h, o)$ on the axis.

Drop perpendiculars PN, $\mathsf{P_1N_1}$.

It is required to prove $\mathsf{AN.AN_1}$ is constant.

The equation of any line $\mathsf{PP_1}$ through B is

$$y = m\,(x - h).$$

Combining this with the equation $y^2 = 4ax$

$$m^2\,(x - h)^2 = 4ax.$$

$$\therefore \quad m^2\,x^2 - 2m^2\,hx + h^2\,m^2 = 4ax$$

$$\text{or} \quad x^2 - \left(2h + \frac{4a}{m^2}\right)x + h^2 = 0. \qquad (1)$$

If the roots of this equation are x_1, x_2, then $x_1 x_2 = h^2$.

But the roots of (*1*) are the abscissae of the extremities of the chord PP_1.

Hence $AN.AN_1 = h^2$, a constant.

98.—Exercises

1. If the ordinates of any two points on the parabola $y^2 = 4ax$ are in the ratio 1 to 2 show that the corresponding abscissae are in the ratio 1 to 4.

Hence show that the square of the ratio of the ordinates of any two points on a parabola is equal to the ratio of the corresponding abscissae.

2. The altitude of a parabolic segment is 8 feet and the length of its base is 14 feet. A straight line drawn across the segment perpendicular to its axis is 7 feet long. How far is it from the vertex of the segment?

3. An arch in the form of a parabolic curve, with vertical axis, is 40 feet across the bottom and the highest point is 12 feet above the horizontal. What is the length of a beam placed horizontally across the arch 3 feet from the top.

4. A parabolic arch has a base 80 feet long and its highest point is 64 feet. Find the height of the arch at a point 10 feet from either end.

5. The cable of a suspension bridge hangs in the form of a parabola. The roadway, which is horizontal and 300 feet long is supported by vertical wires attached to the cable, the longest wire being 90 feet and the shortest wire being 20 feet. Find the length of the supporting wire attached to the roadway 50 feet from the middle.

6. A suspension bridge of 80 foot span is to carry a platform by vertical tie rods at 8 foot intervals. If the height of the piers is 12 feet and the vertex of the parabola in which the chain hangs is 8 feet below the tops of the piers find the length of each tie rod.

7. If p, p^1 are perpendiculars to the y-axis from the extremities of any focal chord of the parabola $y^2 = 4ax$, prove $pp^1 = a^2$.

8. The vertices of a triangle are three points on the parabola $y^2 = 4ax$, whose ordinates are y_1, y_2, y_3. Show that its area is $\frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)$.

9. Find the length of the side of an equilateral triangle inscribed in the parabola $y^2 = 4ax$ if one angular point is at the focus.

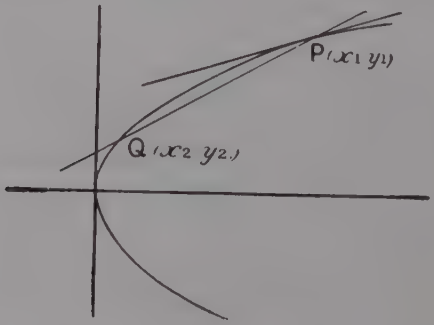
10. Find the equation of a circle which passes through the vertex and the focus of the parabola $y^2 = 8x$ and has its centre on the line $x - y + 2 = 0$.

11. The circle $(x - h)^2 + (y - k)^2 = r^2$ cuts the parabola $y^2 = 4ax$ in four points whose ordinates are y_1, y_2, y_3, y_4.

Prove $y_1 + y_2 + y_3 + y_4 = 0$.

Tangents and Normals

99. To find the equation of the tangent to the parabola $y^2 = 4ax$ in terms of the co-ordinates of the point of contact (x_1, y_1).



Let Q (x_2, y_2) be another point on the parabola.

Then the equation of **PQ** is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}. \qquad (1)$$

Since **P** and **Q** are both on the parabola,

$$y_1^2 = 4ax_1.$$

$$y_2^2 = 4ax_2.$$

Subtracting, $y_1^2 - y_2^2 = 4a(x_1 - x_2)$.

$$\therefore \quad (y_1 - y_2)(y_1 + y_2) = 4a(x_1 - x_2). \qquad (2)$$

Multiplying the terms of (*2*) by the equal fractions in (*1*) $\quad (y - y_1)(y_1 + y_2) = 4a(x - x_1)$. $\qquad$ (*3*)

If now **PQ** rotates about **P** until **Q** coincides with **P**

Then $x_2 = x_1$, and $y_2 = y_1$.

Thus equation (*3*) becomes

$$2y_1(y - y_1) = 4a(x - x_1)$$

$$\text{or} \quad yy_1 - y_1^2 = 2ax - 2ax_1.$$

$$\therefore \quad yy_1 = 2ax - 2ax_1 + y_1^2$$

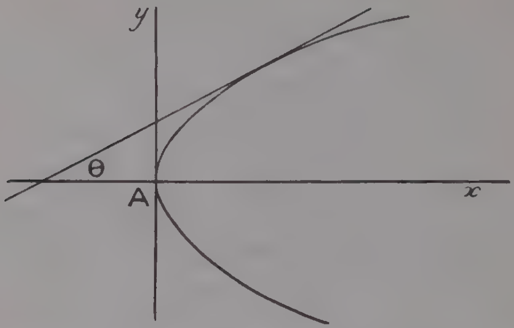
$$\therefore \quad yy_1 = 2ax - 2ax_1 + 4ax_1$$

or $\quad yy_1 = 2a(x + x_1)$, which is the equation of the required tangent.

Cor. 1.—By similar reasoning to that above the equation of the tangent to the parabola $x^2 = 4ay$ at the point $(x_1 y_1)$ is $xx_1 = 2a(y + y_1)$.

Cor. 2.—The tangent to the parabola at the vertex (o, o) is $0 = 2ax$ or $x = 0$. *i.e.* the y-axis.

100. To find the equation of the tangent to the parabola $y^2 = 4ax$ in terms of m, the slope of the tangent.



The tangent may be represented by the line $y = mx + k$ where k is yet to be found.

To find the abscissae of the points where the line $y = mx + k$ cuts the parabola $y^2 = 4ax$, eliminate y by substitution, and

$$(mx + k)^2 = 4ax$$

$$\text{or} \quad m^2x^2 + 2(mk - 2a)x + k^2 = 0.$$

If the line is a tangent, the values of x from this equation are equal to each other.

The condition for equal values of x is

$$m^2k^2 = (mk - 2a)^2$$

$$\text{or} \qquad k = \frac{a}{m}.$$

Thus the equation of the tangent is

$$y = mx + \frac{a}{m}.$$

101. Alternative proofs of the preceding proposition:

[illegible]

(*1*) It has been shown that $yy_1 = 2a(x + x_1)$ is the tangent at the point (x_1, y_1).

Comparing $yy_1 = 2a(x + x_1)$

$$\text{or } y = \frac{2a}{y_1}x + \frac{2ax_1}{y_1}$$

$$\text{and } y = mx + b.$$

$$m = \frac{2a}{y_1}$$

$$\text{or } y_1 = \frac{2a}{m}$$

$$\text{and } b = \frac{2ax_1}{y_1} = \frac{4ax_1}{2y_1} = \frac{(y_1)^2}{2y_1} = \frac{y_1}{2} = \frac{2a}{2m} = \frac{a}{m}$$

$\therefore\ y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$.

(*2*) Since $\left(\frac{2a}{m}\right)^2 = 4a\left(\frac{a}{m^2}\right)$

the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ is on the parabola $y^2 = 4ax$.

Substituting $\frac{a}{m^2}, \frac{2a}{m}$ for x_1, y_1 in the equation of the tangent $yy_1 = 2a(x + x_1)$

$$y \cdot \frac{2a}{m} = 2a\left(x + \frac{a}{m^2}\right)$$

which reduces to $y = mx + \frac{a}{m}$.

The equation of the tangent to the parabola $x^2 = 4ay$ having a slope "m" is $y = mx - am^2$.

102. Two tangents can be drawn to the parabola $y^2 = 4ax$ from any point which is outside the parabola.

The equation of any tangent is

$$y = mx + \frac{a}{m}.$$

This line will pass through the point (x_1, y_1) if

$$y_1 = mx_1 + \frac{a}{m}$$

that is if $m^2x_1 - my_1 + a = 0$.

This being a quadratic gives two values of m, and therefore in general two tangents can be drawn from a given point (x_1, y_1) to a parabola. This equation also gives the slopes of the tangents passing through the point (x_1, y_1).

103. **If two tangents to the parabola $y^2 = 4ax$ intersect at right angles, find the locus of their intersection.**

Let the tangents be $y = mx + \dfrac{a}{m}$ (*1*)

$$\text{and } y = m_1x + \frac{a}{m_1}. \qquad (2)$$

Applying the condition of perpendicularity

$$mm_1 = -1. \tag{3}$$

Subtracting (*2*) from (*1*)

$$0 = x(m - m_1) + a\left(\frac{m_1 - m}{mm_1}\right)$$

$$\text{or } 0 = x - \frac{a}{mm_1}.$$

Using (*3*), $0 = x + a$

$$x = -a$$

i.e. the locus is the directrix.

104. To find the equation of the normal to the parabola $y^2 = 4ax$ at the point (x_1, y_1).

The equation of any line through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1). \tag{1}$$

If this be the normal at (x_1, y_1) it is perpendicular to the tangent $yy_1 = 2a(x + x_1)$

$$\text{or } y = \frac{2a}{y_1}(x + x_1).$$

The condition for perpendicularity is

$$m = -\frac{y_1}{2a}.$$

Substituting this value of m in (*1*) thus making (*1*) the normal, the equation is $y - y_1 = -\dfrac{y_1}{2a}(x - x_1)$

$$\text{or } x - x_1 + \frac{2a}{y_1}(y - y_1) = 0.$$

105.—Exercises

1. Write the equations of the tangent and normal to each of the following parabolas at the point indicated:

(a) $y^2 = 4x$ at point $(1, 2)$.
(b) $x^2 = 8y$ at point $(4, 2)$.
(c) $y^2 = 9x$ at point $(1, 3)$.
(d) $x^2 = 25y$ at point $(5, 1)$.
(e) $x^2 = -4y$ at point $(1, -\frac{1}{4})$.

2. Find the equation of the tangents to the following parabolas, having the inclinations indicated:

(a) $y^2 = 3x$, inclination 30°, 45°, 135°.
(b) $x^2 = 8y$, inclination 60°, 120°, 150°.

3. Find the equations of the tangents to the parabola $y^2 = 4ax$ drawn at the extremities of the latus rectum.

4. Find the equations of the tangents to the parabola $y^2 = 12x$ drawn at the extremities of the latus rectum.

5. Find the equations of the tangents to the parabola $y^2 = 6x$ drawn at the extremities of a focal chord making an angle of 45° with the axis.

6. For what value of K will the line $y = 2x + K$ be a tangent to the parabola $y^2 = 8x$?

7. For what value of p will the line $y = 3x - p$ be a tangent to the parabola $y^2 = 3x$?

8. Find the equations of the tangents to the parabola $y^2 = 4x$ which pass through the point $(1, 5)$.

9. Find the equations of the tangents to the parabola $y^2 = 2x$ which pass through $(-2, -2)$.

10. Find the equation of a tangent to the parabola $y^2 = 8x$ perpendicular to $2x - 3y + 6 = 0$.

11. Find the equation of a tangent to the parabola $y^2 = 8x$ parallel to $3x + 4y + 7 = 0$.

12. Find the equations of the normals to the parabola $y^2 = 20x$ at the extremities of the latus rectum.

13. Find the equations of the normals to the parabola $y^2 = 4x$ drawn at the points of intersection of the line $x + 2y = 6$ and the parabola.

14. Show that the tangent to the parabola $y^2 = 4ax$ at the point (x^1, y^1) is perpendicular to the tangent at the point $\left(\frac{a^2}{x^1}, -\frac{4a^2}{y^1}\right)$.

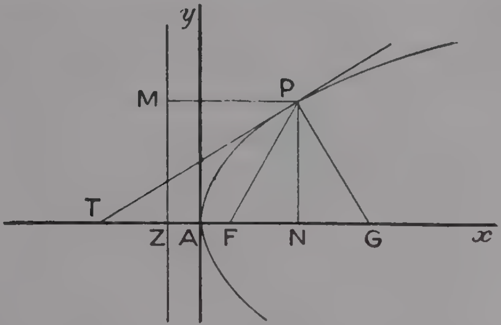
15. Find the equation of the common tangent to the parabolas $y^2 = 4ax$ and $x^2 = 4by$.

16. Find the equation of the common tangent to the parabolas $y^2 = 12x$ and $x^2 = 8y$.

17. Show that the line joining any point in the directrix to the focus of the parabola $y^2 = 4ax$ is perpendicular to the chord of contact of tangents drawn from that point.

18. Prove that the tangent at any point on the parabola $y^2 = 4ax$ will meet the directrix and latus rectum produced, in points equidistant from the focus.

106. At any point on the parabola the tangent bisects the angle between the line from that point to the focus and the perpendicular on the directrix.



Let **P** (x_1, y_1) be any point on the parabola; **PT** the tangent; **PF** the focal distance; **PM** the perpendicular on the directrix.

It is required to prove angles **FPT**, **MPT** equal.

Equation of tangent PT is $yy_1 = 2a(x + x_1)$.

At the point T, $x = \text{AT}$, $y = 0$

$$\therefore\ 0 = 2a(\text{AT} + x_1)$$

$$\text{and AT} = -x_1$$

$$\text{Hence TA} = \text{AN}$$

$$\text{Also FA} = \text{AZ}$$

$$\text{Adding, TF} = \text{ZN} = \text{MP} = \text{PF}$$

$\therefore$ triangle FTP is isosceles

and $\angle \text{FPT} = \angle \text{FTP} = \angle \text{MPT}$;

Hence PT bisects the $\angle$ MPF.

Cor.—This proposition gives the pupil a short method for drawing a tangent to the parabola at any point P.

Measure FT on the axis equal to FP and join PT.

The line PT is the tangent at P.

The line TN is called the **subtangent.**

If PG be the normal at P the line NG is called the **subnormal.**

omit

107. To show that in the parabola the subnormal is constant.

In § 106 the co-ordinates of G are AG, O.

Substituting these values in the equation of the normal

$$x - x_1 + \frac{2a}{y_1}(y - y_1) = 0.$$

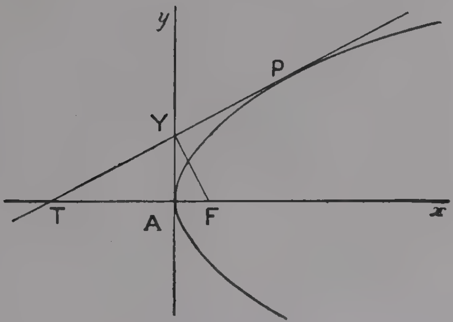
$$\text{AG} - x_1 + \frac{2a}{y_1}(o - y_1) = 0.$$

$$\therefore\ \text{AG} = x_1 + 2a.$$

$$\text{But AN} = x_1$$

$$\therefore\ \text{NG} = 2a, \text{ which is constant.}$$

108. To find the locus of the foot of the perpendicular from the focus of a parabola on the tangent.



The equation of the tangent is $y = mx + \frac{a}{m}$. (1)

The equation of a line through the focus (a, o) is $y = m_1 (x - a)$. (2)

If (2) is perpendicular to (1) then $m_1 = -\frac{1}{m}$

Equation (2) becomes $y = -\frac{1}{m}(x - a)$. (3)

Combining (1) and (3) to obtain the point of intersection, $mx + \frac{a}{m} = -\frac{x}{m} + \frac{a}{m}$

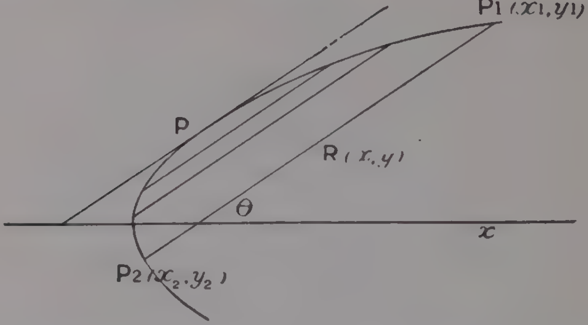
or $(m^2 + 1)x = 0$.

$m^2 + 1$ cannot equal 0.

$\therefore x = 0$.

i.e. Ay, the tangent at the vertex, is the locus of the foot of the perpendicular from the focus on the tangent.

109. To find the locus of the middle points of parallel chords of the parabola $y^2 = 4ax$.



Let the chords be inclined to the axis at an angle θ, **P**$_1$ (x_1, y_1), **P**$_2$ (x_2, y_2) the extremities of one of the chords, and **R** $(x, y,)$ its middle point.

$$x = \frac{x_1 + x_2}{2} \tag{1}$$

$$\text{and } y = \frac{y_1 + y_2}{2}$$

$$\tan\, \theta = \frac{y_1 - y_2}{x_1 - x_2}. \tag{2}$$

Since (x_1, y_1) and (x_2, y_2) are on the parabola.

$$y_1^2 = 4ax_1$$

$$y_2^2 = 4ax_2.$$

Subtracting, $y_1^2 - y_2^2 = 4a(x_1 - x_2)$

or $(y_1 - y_2)(y_1 + y_2) = 4a(x_1 - x_2)$.

Dividing by $2(x_1 - x_2)$

$$\frac{y_1 - y_2}{x_1 - x_2}\left(\frac{y_1 + y_2}{2}\right) = 2a$$

$$y.\ \tan\, \theta = 2a$$

$$\text{or } y = 2a \cot \theta.$$

This is the equation of the locus and is a straight line parallel to the axis.

The locus of middle points of parallel chords of a parabola is called a **diameter.**

Let the chord P_1P_2 (§ 109) move parallel to itself towards P, the point where the diameter cuts the parabola;

Then ultimately the bisected chord becomes the tangent at P.

Hence the tangent at the extremity of a diameter is parallel to the chords which the diameter bisects.

110.—Exercises

1. In the parabola $y^2 = 4x$ find the length of the subtangents at the points (16, 8); $(2, 2\sqrt{2})$; $(3, 2\sqrt{3})$.

2. In the parabola $y^2 = \frac{1}{2}x$, find the length of the subtangents at the points (1, 2); $(\sqrt{2}, 4)$; $(\sqrt{3}, 6)$.

3. Prove by a diagram that the length of the subtangent to the parabola $y^2 = 4ax$ is always equal to twice the abscissae of the point of contact of the tangent.

4. Show that the subtangent of the parabola $y^2 = 4ax$ is bisected at the vertex.

5. For what point on the parabola $y^2 = 4ax$ is the normal equal to the subtangent.

6. Show that the tangent at any point on the parabola $y^2 = 4ax$ meets the directrix and latus rectum at points equidistant from the focus.

7. If PT be a tangent to the parabola $y^2 = 4ax$ at the point P, and FY the perpendicular from the focus on the tangent, prove $FY^2 = AF.FP$.

8. If the ordinate PN at any point P on the parabola $y^2 = 4ax$ meet LT the tangent at the end of the latus rectum in T, then TN equals PF.

9. Prove that tangents to the parabola $y^2 = 4ax$ from any point on the directrix are perpendicular to each other.

10. Two tangents are drawn to the parabola $y^2 = 4ax$ from any point **P** (x_1, y_1). If P_1 and P_2 be the perpendiculars from the focus on these tangents, prove $\mathsf{P}_1.\mathsf{P}_2 = a.\mathsf{PF}$.

11. Find equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (h, k).

12. Find the locus of the middle points of chords through (x_1, y_1) of the parabola $y^2 = 4ax$.

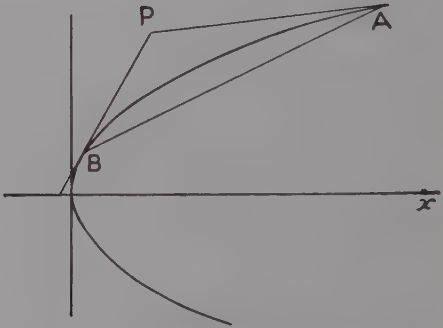
13. Find the locus of the middle points of focal chords of the parabola $y^2 = 4ax$.

14. Show that the normal at one extremity of the latus rectum of a parabola is parallel to the tangent at the other extremity.

15. From any point on the latus rectum of a parabola $y^2 = 4ax$ perpendiculars are drawn to the tangents at its extremities. Show that the line joining the feet of these perpendiculars is a tangent to the parabola.

POLES AND POLARS

111. To find the equation of the chord of contact of tangents drawn from an outside point to the parabola $y^2 = 4ax$.



Let **P** (x_1, y_1) be the given point, and **PA** and **PB** the given tangents. It is required to find the equation of **AB**.

Let (h, k) and (h^1, k^1) represent the co-ordinates of A and B respectively.

The equation of PA is

$$yk = 2a\,(x + h).$$

The equation of PB is

$$yk^1 = 2a\,(x + h^1).$$

Since the point P (x_1, y_1) is on both these lines

$$y_1 k = 2a\,(x_1 + h) \qquad (1)$$

$$\text{and } y_1 k^1 = 2a\,(x_1 + h^1). \qquad (2)$$

The equations (*1*) and (*2*) are the conditions that the points (h, k) and $(h^1\ k^1)$ may lie on the straight line whose equation is

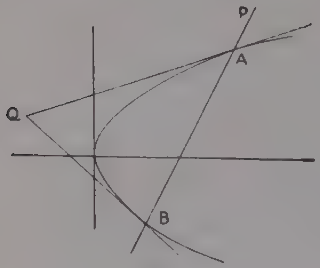
$$y_1 y = 2a\,(x_1 + x)$$

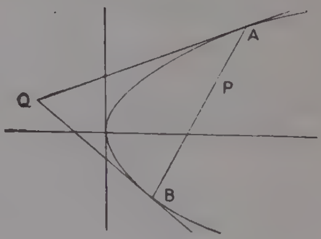
which is the required equation of the chord of contact of the tangents drawn from the point (x_1, y_1).

If the point P approach the parabola and ultimately fall on it, the chord of contact becomes the tangent at P or the tangent at P is the final position of the chord of contact when P approaches the parabola.

This is the reason that the equation of the chord of contact of tangents drawn from an outside point to a parabola is of the same form as the equation of the tangent at a point on the parabola.

112. To find the equation of the polar of P (x_1, y_1) with respect to the parabola $y^2 = 4ax$.





Through **P** draw any straight line cutting the parabola at **A**, **B**. Draw tangents **AQ**, **BQ** intersecting at **Q** (h, k).

It is required to find the locus of **Q**.

By § 111 the equation of **AB** is

$$yk = 2a\,(x + h)$$

And as the co-ordinates of **P** must satisfy this equation,

$$y_1 k = 2a\,(x_1 + h)$$

as h, k are the co-ordinates of any point on the polar of **P**, the required equation is

$$yy_1 = 2a\,(x + x_1)$$

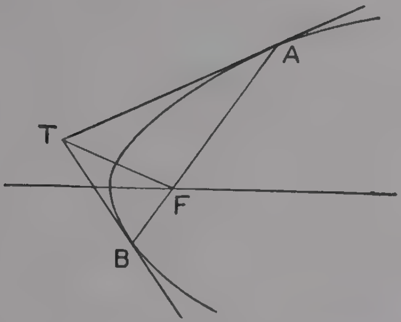
The polar of the focus (a, o) is

$$x + a = o$$

i.e. the directrix; therefore the tangents at the ends of a focal chord intersect on the directrix, or the directrix is the locus of the intersection of tangents at the extremities of focal chords.

The equation of the polar of the point **P** (x_1, y_1) without the parabola $y^2 = 4ax$ is the same as that of the chord of contact of tangents drawn from **P** to the parabola; or tangents drawn from **P** touch the parabola at the points where it is cut by the polar of **P**.

113. **In the parabola the tangents at the extremities of any focal chord are at right angles to each other.**



Let **T** $(-a, k)$ be a point on the directrix where the tangents **AT**, **BT** at the ends of the focal chord **AFB** intersect.

It is required to prove that **AT** and **BT** are at right angles.

Let $y = mx + \frac{a}{m}$ be either of these tangents.

Since it passes through $(-a, k)$

$$k = -ma + \frac{a}{m}$$

$$\text{or } m^2 + \frac{km}{a} - 1 = 0$$

which is the relation connecting the m's of **AT** and **BT**.

If m_1 and m_2 be the roots of this equation.

Then $m_1 . m_2 = -1$. *i.e.* AT and BT are at right angles.

114. In the parabola the focal chord is at right angles to the line joining its pole to the focus.

Using the figure of § 113.

Let AFB be any focal chord, T $(-a, k)$ its pole. It is required to prove AB at right angles to FT.

The equation of AB, which is the polar of $(-a, k)$ is

$$yk = 2a(x - a)$$

The equation of TF, through (a, o) $(-a, k)$ is

$$\frac{y}{k} + \frac{x-a}{2a} = 0.$$

and these lines satisfy the condition of perpendicularity.

115. If the polar of P passes through Q, the polar of Q passes through P.

Let (x_1, y_1) (x_2, y_2) be the co-ordinates of P, Q respectively.

The polar of P with respect to parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1)$$

Since this line passes through Q

$$y_1y_2 = 2a(x_1 + x_2)$$

This proves that x_1, y_1, is on the line

$$yy_2 = 2a(x + x_2)$$

and therefore P is on the polar of Q.

Cor. If the point Q moves along the polar of P, the polar of Q changes its position but always passes through P.

∴ If the pole moves along a straight line, its polar turns about the pole of that line.

116. To find the pole of the straight line $Ax + By + C = 0$ with respect to the parabola $y^2 = 4ax$.

Let (x_1, y_1) be the co-ordinates of the pole.

The equation of the polar of (x_1, y_1) is

$$yy_1 = 2a\,(x + x_1). \qquad (1)$$

This equation must be the same as

$$Ax + By + C = 0$$

or

$$By = -Ax - C. \qquad (2)$$

Comparing (*1*) and (*2*)

$$\frac{y_1}{B} = \frac{2a}{-A} = \frac{2ax_1}{-C}.$$

Therefore $x_1 = \dfrac{C}{A}$, $y_1 = \dfrac{-2aB}{A}$.

The pole is $\left(\dfrac{C}{A}, -\dfrac{2aB}{A}\right)$.

117.—Exercises

1. Find the polar of the point:

- (*a*) (3, 5) with respect to $y^2 = 4x$.
- (*b*) (2, 3) with respect to $y^2 = 8x$.
- (*c*) (3, − 1) with respect to $y^2 = 6x$.

2. Find the pole of the straight line:

- (*a*) $2x + 3y = -17$ with respect to $y^2 = 2x$.
- (*b*) $3x - y = 6$ with respect to $y^2 = 8x$.
- (*c*) $6x + 2y = 7$ with respect to $y^2 = 6x$.

3. Find the pole of the line $\frac{x}{a} + \frac{y}{b} = 1$ with respect to the parabola $y^2 = 4px$.

4. Show that the polars of all points on a line parallel to the axis of the parabola are parallel to one another.

5. Write down the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$. What are the co-ordinates of its pole? What is the locus of the pole.

6. (*a*) Show that (2, 4) is on the parabola $y^2 = 8x$.
   (*b*) Write the equation of the tangent to the parabola at this point.
   (*c*) Show that the point (6, 4), is on this tangent.
   (*d*) Write the equation of the polar of (6, 4).

Plot the above on squared paper.

7. Find the equation and length of the chord of contact of tangents to $y^2 = 4x$ from the point (2, 3).

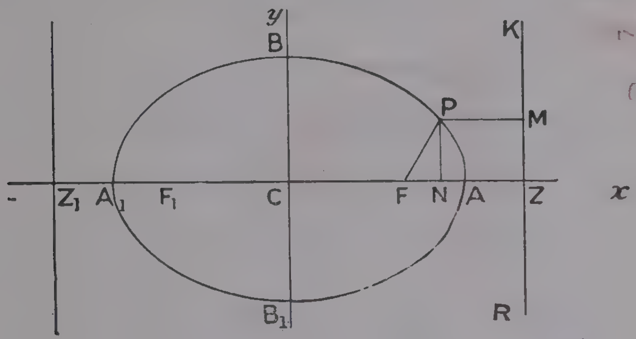
8. If R is the middle point of a chord PQ of the parabola $y^2 = 4ax$. Show that the polar of R is parallel to PQ.

CHAPTER V

THE ELLIPSE

118. **An Ellipse** is the locus of a point which moves so that its distance from a fixed point, called the focus, bears a constant ratio, which is less than unity, to its distance from a fixed straight line, called the directrix.

119. To find the equation of the ellipse.



Let F be the focus and KR the directrix.

Draw FZ perpendicular to KR.

Divide FZ internally at A and externally at A_1

$$\text{So that } \frac{FA}{AZ} = e, \text{ or } FA = e.AZ \qquad (1)$$

$$\text{and } \frac{A_1F}{A_1Z} = e, \text{ or } A_1F = e.A_1Z. \qquad (2)$$

By definition A and A_1 are points on the locus.

Bisect A_1A in C; let $A_1A = 2a$.

Adding (*1*) and (*2*),

$$\mathrm{FA} + \mathrm{A_1F} = e\,(\mathrm{AZ} + \mathrm{A_1Z})$$

$$\therefore\ 2\mathrm{CA} = 2e.\mathrm{CZ}$$

$$\text{or } \mathrm{CZ} = \frac{a}{e} \qquad (3)$$

Subtracting (*1*) from (*2*),

$$\mathrm{A_1F} - \mathrm{FA} = e\,(\mathrm{A_1Z} - \mathrm{AZ})$$

$$\therefore\ \mathrm{A_1A} - 2\mathrm{FA} = e.\mathrm{A_1A}$$

$$\therefore\ 2\mathrm{CF} = 2e.\mathrm{CA}$$

$$\text{or } \mathrm{CF} = ae \qquad (4)$$

Let **C** be taken as the origin, **CZ** as the axis of **X**, and **BCB$_1$** which is perpendicular to **CZ** as the axis of y.

Let **P** (x,y) be any point on the locus.

Draw **PN** perpendicular to **CZ** and **PM** perpendicular to **KR**.

Then **CN** $= x$, **PN** $= y$, the co-ordinates of **F** and **Z** are (ae, o) and $\left(\frac{a}{e}, o\right)$.

By definition of the ellipse,

$$\frac{\mathrm{PF}}{\mathrm{PM}} = e$$

$$\text{or } \mathrm{PF}^2 = e^2.\mathrm{PM}^2 = e^2.\mathrm{NZ}^2$$

$$\therefore\ \mathrm{FN}^2 + \mathrm{PN}^2 = e^2.\mathrm{NZ}^2$$

But **FN** $=$ **CN** $-$ **CF** and **NZ** $=$ **CZ** $-$ **CN**

$$\therefore\ (x - ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$$

$$\text{or } x^2(1-e^2)+y^2=a^2(1-e^2)$$

$$\text{or } \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1 \qquad (5)$$

which is the equation of the ellipse.

Putting $a^2(1-e^2)=b^2$, equation (*5*) takes the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (6)$$

This is the form in which the equation is usually written.

Equation (*3*) gives the distance from the centre to the directrix, $\mathrm{CZ}=\frac{a}{e}$

Equation (*4*) gives the distance from the centre to the focus, $\mathrm{CF}=ae$.

The square of the eccentricity, $e^2=\frac{a^2-b^2}{a^2}$ is obtained from $a^2(1-e^2)=b^2$.

The points A and A_1 are called the **vertices** of the ellipse. C is the **centre**, AA_1 is called the **major axis** and BB_1 the **minor axis**.

120. In deducing the equation of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

the focus and directrix were placed in convenient positions in order to get the equation in the most suitable form from which to study the properties of the ellipse. In the following examples the focus is taken as any point and the directrix as any straight line.

To find the equation of the ellipse whose focus, directrix and eccentricity are given.

(*1*) Let the focus be the point (*4*, *5*), the directrix the line $2x + 3y + 4 = 0$ and the eccentricity $\frac{6}{7}$.

Then by definition, if (x, y) is any point on the locus,

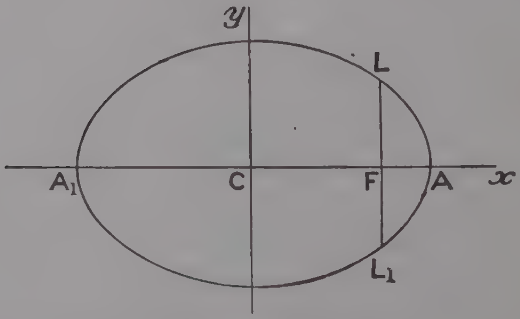
$$(x - 4)^2 + (y - 5)^2 = \left(\frac{6}{7}\right)^2 \frac{(2x + 3y + 4)^2}{13}$$

(2) The equation of the ellipse whose focus is the point (h, k), whose directrix is the line $\mathsf{A}x + \mathsf{B}y + \mathsf{C} = 0$ and whose eccentricity is e

$$\text{is } (x - h)^2 + (y - k)^2 = e^2 \frac{(\mathsf{A}x + \mathsf{B}y + \mathsf{C})^2}{\mathsf{A}^2 + \mathsf{B}^2}$$

Unless otherwise stated the equation of the ellipse will be assumed to be of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

121. To find the length of a latus rectum of the ellipse.



LFL₁ is a latus rectum of the ellipse

Substitute $x = ae$ in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

then $$\frac{a^2e^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\text{or } \frac{y^2}{b^2} = 1 - e^2.$$

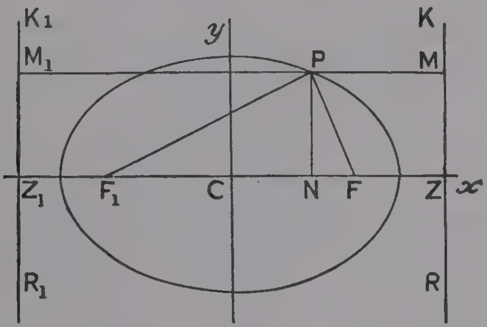
$$= 1 - \frac{a^2 - b^2}{a^2}$$

$$= \frac{b^2}{a^2}$$

$$y^2 = \frac{b^4}{a^2} \text{ or } y = \frac{b^2}{a}$$

$\therefore$ length of latus rectum $= 2\frac{b^2}{a}\cdot$

122. **The sum of the focal distances of any point on an ellipse is constant and equal to 2a.**



Let P (x, y) be any point on the ellipse, F and F_1 the foci, KR and K_1R_1 the directrices. It is required to prove $PF + PF_1 = 2a$.

$$\begin{aligned} \mathsf{PF} &= e\cdot\mathsf{PM} = e\cdot\mathsf{NZ} \\ &= e\,(\mathsf{CZ} - \mathsf{CN}) \\ &= e\left(\frac{a}{e} - x\right) = a - ex. \end{aligned}$$

$$\begin{aligned} \mathsf{PF_1} &= e\cdot\mathsf{M_1P} = e\cdot\mathsf{Z_1N} \\ &= e\,(\mathsf{Z_1C} + \mathsf{CN}) \\ &= e\left(\frac{a}{e} + x\right) = a + ex. \end{aligned}$$

$$\begin{aligned} \therefore\ \mathsf{PF} + \mathsf{PF_1} &= (a - ex) + (a + ex) \\ &= 2a. \end{aligned}$$

This property leads to another definition of the ellipse.

An ellipse is the locus of a point which moves so that the sum of its distances from two fixed points is constant.

Let the constant sum be $2a$, and the distances between the two fixed points be $2ae$. Let the middle point of the line joining the fixed points be the origin and let this line and a line perpendicular to it be the axes. Then from the definition above

$$\sqrt{(x + ae)^2 + y^2} + \sqrt{(x - ae)^2 + y^2} = 2a.$$

Rationalizing, this equation becomes

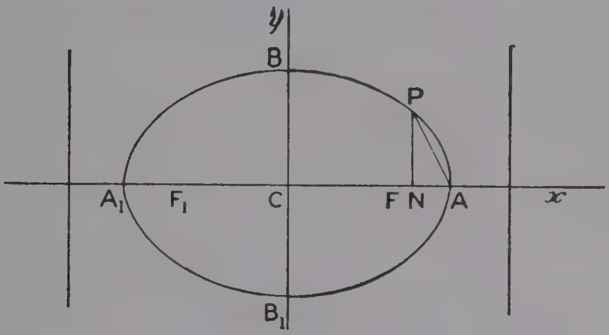
$$y^2 + x^2\,(1 - e^2) = a^2\,(1 - e^2)$$

or $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2\,(1 - e^2)} = 1$ which is equation (5) obtained in § 119.

This gives a method of describing the ellipse mechanically.

If the ends of a thread are fastened at two fixed points F and F_1 then a pencil moved about so as to keep the thread always stretched will describe an ellipse whose foci are F and F_1 because $PF + PF_1$ will be constant.

123 To trace the form of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from its equation.



(*1*) If $x = 0$, $y = \pm b$; if $y = 0$, $x = \pm a$. Therefore the curve cuts the x axis at $\pm a$ and the y axis at $\pm b$.

(*2*) $x^2 = \frac{a^2}{b^2}(b^2 - y^2)$. Hence the value of y cannot be greater, numerically, than $\pm b$, or otherwise x^2 would be negative. Similarly x cannot be greater, numerically, than $\pm a$. Therefore an ellipse is a curve which is limited in all directions.

(*3*) $y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$. Therefore as x increases from o to a, y decreases from $\pm b$ to o.

(*4*) For any particular value of x the values of y are equal with opposite signs. The axis of x therefore divides the curve into two similar and equal parts. Similarly the curve is symmetrical with respect to the axis of y.

(*5*) Let the straight line $y = mx + c$ cut the ellipse. The abscissae of the points of intersection are obtained from the equation $\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$, a quadratic giving two values of x. Hence a straight line can cut an ellipse in only two points, which are real, coincident or imaginary.

(*6*) If P (x, y) be any point on the curve,

$$\tan \text{PAN} = \frac{\text{PN}}{\text{AN}} = \frac{y}{a - x} = \frac{y^2}{y\,(a - x)} = \frac{b^2}{a^2} \cdot \frac{a^2 - x^2}{y\,(a - x)}$$

$$= \frac{b^2}{a^2} \cdot \frac{a + x}{y}$$

Ultimately when P moves along the curve indefinitely close to A the value of $y = 0$ and $x = a$.

$$\text{Then } \tan \text{PAN} = \frac{b^2}{a^2} \cdot \frac{a + a}{0} = \infty$$

$\therefore$ angle PAN in the limit $= 90°$.

Hence the curve cuts the axis of x at right angles at the point A.

Similarly it cuts the axis of y at right angles at the point B.

On account of the symmetry of the curve it will cut the axes of x and y at right angles at the points A_1 and B_1 respectively.

Since the axis of y divides the curve into two similar and equal parts it follows that if on the axis of x the points F_1 and Z_1 be taken such that $\mathsf{CF}_1 = \mathsf{CF}$ and $\mathsf{CZ}_1 = \mathsf{CZ}$ the point F_1 will also be a focus of the curve and the line through Z_1 perpendicular to CZ_1 will be the corresponding directrix.

124. Every chord through the centre of an ellipse is there bisected.

Let (x_1, y_1) be any point on the curve. The co-ordinates (x_1, y_1) will satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$\therefore \frac{(x_1)^2}{a^2} + \frac{(y_1)^2}{b^2} = 1$$

$$\text{also } \frac{(-x_1)^2}{a^2} + \frac{(-y_1)^2}{b^2} = 1$$

i.e.—the point $(-x_1, -y_1)$ is also on the curve.

But the points (x_1, y_1) and $(-x_1, -y_1)$ are on a straight line through the origin and are equidistant from the origin,

$$\text{since } \sqrt{(x_1)^2 + (y_1)^2} = \sqrt{(-x_1)^2 + (-y_1)^2}.$$

Hence the origin bisects every chord which passes through it, and is therefore called the centre of the ellipse.

125. Examples

(*1*) **Find the equation of an ellipse whose focus is (1, 0), directrix $\mathbf{x - 4 = 0}$ and eccentricity $\frac{1}{2}$.**

From the definition of the ellipse,

$$\sqrt{(x-1)^2+y^2} = \tfrac{1}{2}(x-4)$$

or reducing

$$3x^2+4y^2=12.$$

(*2*) **Find the eccentricity, the length of the latus rectum, the co-ordinates of the foci, the semi-axes of the ellipse $9x^2+25y^2=225$.**

This equation can be written in the form:

$$\frac{x^2}{\frac{225}{9}}+\frac{y^2}{\frac{225}{25}}=1$$

$$\text{or } \frac{x^2}{5^2}+\frac{y^2}{3^2}=1$$

$\therefore$ length of semi-axes is 5 and 3,

$$e^2=\frac{a^2-b^2}{a^2}=\frac{25-9}{25}=\frac{16}{25}$$

$$\therefore e=\tfrac{4}{5}$$

The latus rectum $= 2\dfrac{b^2}{a} = 2\times\tfrac{9}{5} = 3\tfrac{3}{5}$

$ae = 5\times\tfrac{4}{5}=4$; $\therefore$ co-ordinates of foci $= (\pm 4, 0)$

(*3*) **Find the centre, vertices, eccentricity, and foci of the ellipse $4x^2+9y^2+16x-18y-11=0$**

This equation can be written in the form

$$(x^2+4x+4)+\tfrac{9}{4}(y^2-2y+1)=\tfrac{11}{4}+\tfrac{25}{4}=9$$

$$\text{or } \frac{(x+2)^2}{3^2}+\frac{(y-1)^2}{2^2}=1$$

co-ordinates of centre $= (-2, 1)$ and the lengths of the semi-axes are 3 and 2.

$$e^2 = \frac{a^2 - b^2}{a^2} = \tfrac{5}{9} \text{ or } e = \tfrac{1}{3}\sqrt{5}$$

$a = 3$, $\therefore$ co-ordinates of vertices are $(3 - 2)$, 1 and $(-3 - 2)$, 1 or $(1, 1)$ and $(-5, 1)$

$ae = 3 \times \frac{1}{3}\sqrt{5} = \sqrt{5}$. $\therefore$ co-ordinates of foci are $(-2 \pm \sqrt{5})$ and 1.

In this problem the major axis is parallel to the axis of x.

126. Exercises

1. Find the equation of the ellipse whose focus is $(1, 0)$, directrix $x + y = 0$ and eccentricity $\frac{1}{2}$.

2. Find the eccentricity, the length of the latus rectum and the co-ordinates of the foci of each of the ellipses.

$$4x^2 + 9y^2 = 36 \text{ and } 9x^2 + 4y^2 = 36.$$

3. Given the ellipse $3x^2 + 4y^2 = 2$, find its semi-axes, eccentricity and foci.

4. Find the centre, vertices, eccentricity and foci of the ellipse $16x^2 + 9y^2 - 16x + 6y - 139 = 0$.

5. Find the eccentricity of the ellipse $2x^2 + y^2 = 3x$, also its foci and directrices.

6. Find the equation of the ellipse when the origin is at the left hand vertex and the major axis lies along the x axis.

7. Find the equation of the ellipse when the origin is taken at the lower extremity of the minor axis and the minor axis lies along the axis of y.

8. Find the semi-axes of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ so that it shall pass through the points $(2, 3)$ and $(-1, -4)$.

9. Show that the equation $4x^2 + 2y^2 = 6x$ represents an ellipse whose eccentricity is $\frac{1}{\sqrt{2}}$ and whose origin is at an extremity of the minor axis.

10. Find the equation of an ellipse if its axes are 8 and 4, its centre is $(2, -3)$ and its major axis is parallel to the x axis.

11. Find the equation of the ellipse which has the point $(-1, 1)$ for focus, the line $4x - 3y = 0$ for directrix, and whose eccentricity is $\frac{5}{6}$.

12. Find the equation of the ellipse if its axes are $\frac{2}{5}$ and $\frac{1}{3}$, its centre is at $(1, -1)$ and its major axis is parallel to y axis.

127. Exercises

1. Find the eccentricities, the co-ordinates of the foci, and the lengths of the latus recta of the following ellipses:

$$(a)\ 2(x-1)^2 + 3(y+2)^2 = 1.$$
$$(b)\ 8(x-1)^2 + 6(y+1)^2 = 1.$$
$$(c)\ 4x^2 + 5y^2 - 8x - 20y = 0.$$

2. The vertices of an ellipse are $(\pm 6, 0)$ and its foci are $(\pm 4, 0)$. Find its equation.

3. Find the equation of an ellipse when the vertices are $(\pm 4, 0)$ and one focus is $(2, 0)$.

4. Find the equation of an ellipse when the vertices are $(0, 2)$ and $(0, -4)$ and one focus is at the origin.

5. Find the equation of the ellipse whose foci are $(0, \pm 3)$ and major axis is 10.

6. Find the equation of an ellipse when its centre is at the origin, one focus is at the point $(-4, 0)$ and the minor axis is equal to 6.

7. The centre of an ellipse is at $(-2, 3)$ and its major axis of 8 units in length is parallel to the y axis. Its eccentricity is $\frac{1}{3}$. Find its equation.

8. Find the equation of an ellipse whose eccentricity is $\frac{1}{3}$, the ordinate at the focus 4, the centre being at the origin, and the major axis lying along the x axis.

9. The straight lines joining a point P on the ellipse to the extremities of the minor axis meets the major axis in q and t. Prove that $\mathsf{C}q.\mathsf{C}t = \mathsf{CA}^2$ where C is the centre of the ellipse and A one of the extremities of the major axis.

10. Find the equation of an ellipse and its eccentricity if the foci lie halfway between the centre and the vertices, the centre being at the origin and the major axis lying along the x axis.

11. Find the eccentricity of an ellipse if the straight line connecting the positive ends of the axes is parallel to the straight line joining the centre to the upper end of the ordinate at the left hand focus.

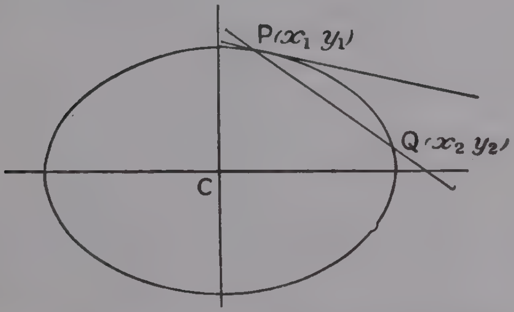
12. Find the equation and the eccentricity of an ellipse if the ordinate at the focus is one third the minor axis, the centre being at the origin and the major axis lying along the x axis.

13. Find the equation of the ellipse whose latus rectum is $\frac{9}{2}$, eccentricity $\frac{\sqrt{7}}{4}$, the centre being at the origin and the major axis lying along the x axis.

14. The major axis of an ellipse is divided into two parts equal to the focal distances of a point **P** on the ellipse. Prove that the distance of the point of division from either end of the minor axis is equal to the distance of **P** from the centre.

TANGENTS AND NORMALS

128. To find the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, in terms of the co-ordinates of the point of contact (x_1, y_1).



Let **Q** (x_2, y_2) be another point on the ellipse.

Then the equation of **PQ** is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} \tag{1}$$

Since **P** and **Q** are both on the ellipse,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

and $$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1.$$

subtracting $\frac{1}{a^2}(x_1^2 - x_2^2) + \frac{1}{b^2}(y_1^2 - y_2^2) = 0.$

or $$\frac{(x_1 - x_2)(x_1 + x_2)}{a^2} = -\frac{(y_1 - y_2)(y_1 + y_2)}{b^2} \tag{2}$$

Multiplying the terms of (*2*) by the equal fractions in (*1*)

$$\frac{(x_1 + x_2)(x - x_1)}{a^2} = -\frac{(y_1 + y_2)(y - y_1)}{b^2} \tag{3}$$

If now **PQ** rotates about **P** until **Q** coincides with **P** then $x_2 = x_1$, and $y_2 = y_1$ and equation (*3*) becomes

$$\frac{2x_1(x - x_1)}{a^2} = -\frac{2y_1(y - y_1)}{b^2}$$

or $$\frac{xx_1}{a^2} - \frac{x_1^2}{a^2} = \frac{y_1^2}{b^2} - \frac{yy_1}{b^2}$$

or $$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

$\therefore \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ is the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1). (*4*)

Cor. 1.—The co-ordinates at the extremities of the major axis are (a, o) and $(-a, o)$ respectively. Substituting these co-ordinates in *(4)* the tangents at these points are $x = a$ and $x = -a$. Hence the tangents at the extremities of the major axis are perpendicular to that axis and are parallel to the minor axis. Similarly the tangents at the extremities of the minor axis are parallel to the major axis.

Cor. 2—The tangent at the point (x_1, y_1) is parallel to the tangent at the point $(-x_1, -y_1)$ since the tangent at $(-x_1, -y_1)$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = -1$. These two points $(x_1, y_1), (-x_1, -y_1)$ lie on a straight line through the centre of the ellipse. Hence the tangents at the extremities of any chord passing through the centre of an ellipse are parallel.

129. To find the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in terms of m, the slope of the tangent.

The tangent may be represented by the line $y = mx + c$ where c is yet to be found. To find the abscissae of the points where the line $y = mx + c$ cuts the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, eliminate y by substitution giving

$$\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1.$$

or $x^2(b^2 + a^2m^2) + 2mca^2x + a^2(c^2 - b^2) = 0$

If the line is a tangent, the values of x from this equation are equal to each other.

The condition for equal values of x is

$$a^2(c^2 - b^2)(b^2 + a^2m^2) = m^2c^2a^4$$

or $$c^2 = m^2a^2 + b^2$$

or $$c = \pm\sqrt{m^2a^2 + b^2}$$

Thus the equation of the tangent is

$$y = mx \pm \sqrt{m^2a^2 + b^2}$$

The double sign before the radical indicates that there are two tangents to the ellipse for every value of m, that is, there are two tangents parallel to any given straight line. These parallel tangents are equidistant from the centre of the ellipse.

Alternative proof of the preceding proposition:

It has been shown that $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ is the tangent at the point (x_1, y_1)

Comparing $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

and $$y = mx + c$$

or $$mx - y = -c$$

it gives $$\frac{m}{\frac{x_1}{a^2}} = \frac{-1}{\frac{y_1}{b^2}} = \frac{-c}{1}$$

Hence $$c = \frac{-ma}{\frac{x_1}{a}} = \frac{b}{\frac{y_1}{b}} = \pm\frac{\sqrt{m^2a^2 + b^2}}{\sqrt{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}}} = \pm\sqrt{m^2a^2 + b^2}$$

$\therefore$ equation becomes $y = mx \pm \sqrt{m^2a^2 + b^2}$.

130. Two tangents can be drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from any point which is outside the ellipse.

The line whose equation is $y = mx + \sqrt{m^2a^2 + b^2}$ is a tangent to the ellipse for all values of m.

This tangent will pass through the particular point (x_1, y_1) if $y_1 = mx_1 + \sqrt{m^2a^2 + b^2}$.

or if $(y_1 - mx_1)^2 = m^2a^2 + b^2$

or if $m^2 (x_1^2 - a^2) - 2mx_1y_1 + y_1^2 - b^2 = 0$.

This is a quadratic equation which gives the directions of those tangents which pass through the point (x_1, y_1). A quadratic equation has two roots, therefore two tangents to the ellipse will pass through any point (x_1, y_1).

131. To find the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1).

The equation of any line through (x_1, y_1)

$$\text{is} \quad y - y_1 = m(x - x_1) \tag{1}$$

If this be the normal at (x_1, y_1) it is perpendicular to the tangent $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

The condition for perpendicularity is

$$m. \frac{x_1}{a^2} - \frac{y_1}{b^2} = 0.$$

$$\text{or} \qquad m = \frac{y_1}{b^2} \cdot \frac{a^2}{x_1}$$

Substituting this value of m in (*1*) thus making (*1*) the normal, the equation is

$$y - y_1 = \frac{y_1}{b^2} \cdot \frac{a^2}{x_1} (x - x_1)$$

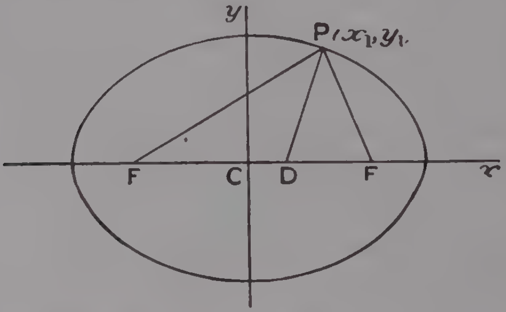
$$\text{or} \quad \frac{b^2}{y_1}(y - y_1) = \frac{a^2}{x_1}(x - x_1)$$

$$\text{or} \quad \frac{a^2}{x_1}x - \frac{b^2}{y_1}y = a^2 - b^2$$

This equation is frequently written in the form

$$\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{\frac{y_1}{b^2}}$$

132. In the ellipse the normal bisects the angle between the focal distances.



Let the normal at P (x_1, y_1) cut the major axis in D. Then the co-ordinates of D are (CD, 0). Substituting these in the equation of the normal it becomes

$$\frac{a^2}{x_1} \cdot \text{CD} = a^2 - b^2$$

$$\text{or} \qquad \text{CD} = \frac{a^2 - b^2}{a^2} \cdot x_1 = e^2 \cdot x_1$$

Hence $\quad \mathsf{F_1D} = ae + e^2x_1$

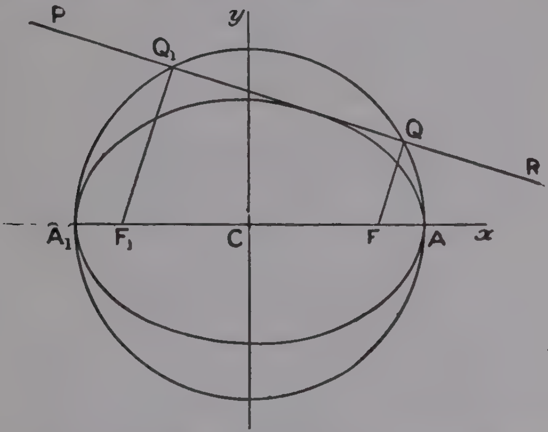
and $\quad \mathsf{DF} = ae - e^2x_1$

$$\therefore \frac{\mathsf{F_1D}}{\mathsf{DF}} = \frac{ae + e^2x_1}{ae - e^2x_1} = \frac{a + ex_1}{a - ex_1} = \frac{\mathsf{F_1P}}{\mathsf{PF}}$$

Hence **PD**, the normal, bisects the angle between the focal distances.

This property gives a short method for drawing the tangent and the normal at any point on the ellipse.

133. **In the ellipse the locus of the foot of the perpendicular dropped from either focus on a tangent is a circle on the axis major as diameter.**



Let **PR** be a tangent at any point and **FQ**, **F₁Q₁** perpendiculars from the foci on **PR**.

Let the equation of **PR** be

$$y = mx + \sqrt{m^2a^2 + b^2} \qquad (1)$$

The line **FQ** through the focus $(ae, 0)$ and perpendicular to **PR** is

$$y = -\frac{1}{m}(x - ae) \qquad (2)$$

Form (*1*) $y - mx = \sqrt{m^2a^2 + b^2}$ (*3*)

Form (*2*) $my + x = ae = \sqrt{a^2 - b^2}$ (*4*)

If we eliminate m from (*3*) and (*4*) the resulting equation is the locus of the foot of the perpendiculars from the focus (ae, o) on any tangent to the ellipse.

Squaring (*3*), $y^2 - 2mxy + m^2x^2 = m^2a^2 + b^2$

Squaring (*4*), $m^2y^2 + 2mxy + x^2 = a^2 - b^2$

Adding $y^2(1 + m^2) + x^2(1 + m^2) = a^2(1 + m^2)$

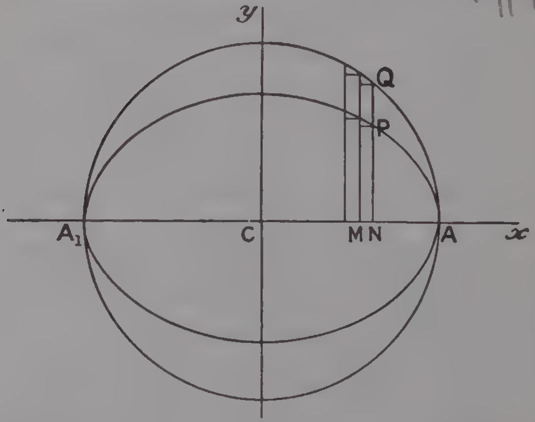
or $x^2 + y^2 = a^2$ which is the equation of a circle on the major axis as diameter.

The same equation is obtained if the line F_1Q_1 through the focus $(-ae, o)$ is considered.

This circle is called the **auxiliary** circle of the ellipse.

134. The area of the ellipse is π ab.

πab



In the ellipse $y = \frac{b}{a}\sqrt{a^2 - x^2}$; in the auxiliary circle $y = \sqrt{a^2 - x^2}$.

Hence **CN** being the x for both **P** and **Q**, **PN** $= \frac{b}{a}$ **QN**; and therefore rectangle **PM**: rectangle **QM** $= b : a$.

This proportion holds for all such corresponding rectangles. Therefore the sum of all such rectangles **PM** is to the sum of all such rectangles **QM** as b is to a.

But if the number of these rectangles be increased indefinitely, their width being indefinitely diminished, their sums become the areas of the ellipse and circle respectively.

Hence, area of ellipse: area of circle $= b : a$;

$$\therefore \text{ area of ellipse} = \frac{b}{a} \cdot \pi a^2$$

$$= \pi ab.$$

135. In the ellipse the locus of the point of intersection of tangents which are at right angles to each other is the circle $x^2 + y^2 = a^2 + b^2$.

The line whose equation is

$$y = mx + \sqrt{m^2a^2 + b^2} \qquad (1)$$

is a tangent to the ellipse for all values of m.

This equation gives us the directions of the tangents which pass through the point (x, y).

Rationalizing (*1*) it becomes

$$m^2 (x^2 - a^2) - 2mxy + y^2 - b^2 = 0 \qquad (2)$$

Let m_1 and m_2 be the roots of (*2*)

If the tangents are at right angles, $m_1m_2 = -1$

The product of the roots of (*2*) is

$$\frac{y^2 - b^2}{x^2 - a^2}$$

Therefore $$\frac{y^2 - b^2}{x^2 - a^2} = -1$$

or $x^2 + y^2 = a^2 + b^2$, which is the equation of a circle.

The required locus is therefore a circle.

This circle is called the **director circle** of the ellipse.

136.—Exercises

1. Determine c so that $y + x = c$ may be a tangent to the ellipse $2x^2 + 3y^2 = 1$.

2. Find the tangent and normal to the ellipse $3x^2 + 4y^2 = 12$ at the end of the latus rectum in the first quadrant.

3. Find the equations of the tangents to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which makes an angle of 60° with an axis of x.

4. Find the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which cut off equal intercepts on the axes.

5. Find the equations of the tangents to the ellipse $3x^2 + 4y^2 = 12$ which cut off equal intercepts on the axes.

6. Show that the line $y = x + \sqrt{\frac{5}{6}}$ touches the ellipse $2x^2 + 3y^2 = 1$.

7. Find the axes of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ so that the ellipse may pass through the point (6, 4) and have the line $3x + 8y = 50$ for tangent at that point.

8. Find b so that $y = x + b$ may touch $\frac{x^2}{16} + \frac{y^2}{4} = 1$, and $y = 2x + b$ may touch $\frac{x^2}{36} + \frac{y^2}{25} = 1$.

9. Find the locus of the middle point of that part of a tangent to an ellipse which is intercepted between the tangents at the vertices of the ellipse.

10. Two tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are such that the product of the tangents of the angles they make with the axis major is $-\frac{b^2}{a^2}$. Show that they intersect on the ellipse $\frac{x^2}{2a^2} + \frac{y^2}{2b^2} = 1$.

11. The normal at **P** meets the axes of an ellipse at **G** and g. Prove that $\mathsf{PG}.\mathsf{P}g = \mathsf{SP}.\mathsf{S}_1\mathsf{P}$ where **S**, S_1 are the foci.

12. A line through **C** (the centre) parallel to the tangent at **P** on an ellipse meets the focal distances of **P** in d and d_1. Prove that $\mathsf{P}d = \mathsf{P}d_1 = a$ (the semi-major axis.)

13. **P** is any point on the ellipse. If $\mathsf{FP} = r$ and the perpendicular from **F** on the tangent at $\mathsf{P} = t$, show that $\frac{b^2}{t^2} = \frac{2a}{r} - 1$.

14. If any ordinate **MP** be produced to meet the tangent at the end of the latus rectum through the focus **F** in **Q**. Prove that the ordinates of **Q** is equal to the distance **FP**.

15. Find the equations of the tangents to the ellipse which are parallel to the line $\frac{x}{a} + \frac{y}{b} = 1$.

16. If the normal at any point **P** on an ellipse cut the major axis in **K**, show that, for different positions of **P**, the locus of the middle point of **PK** is an ellipse.

17. Find the common tangents to the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

18. **P** is any point on the auxiliary circle; the perpendiculars from **F**, F_1 on the tangent at **P** meet the ellipse at q, q_1. Show $\mathsf{P}q$, $\mathsf{P}q_1$ are tangents to the ellipse.

19. If normals be drawn at the extremities of any focal chord of an ellipse, prove that a line through their intersection parallel to the major-axis will bisect the chord.

20. If the perpendicular from the centre **C** of an ellipse on the tangent at any point **P** meet the focal distance **SP**, produced if necessary, in **R** prove that the locus of **R** will be a circle.

21. **C** is the centre of an ellipse and **A** is an extremity of the major axis. **CY** and **AZ** are drawn perpendicular to the tangent at **P**. **N** is the foot of the perpendicular from **P** on the major axis. Prove $CA \cdot AZ = CY \cdot AN$.

22. A line is drawn parallel to the minor-axis of an ellipse midway between a focus and the corresponding directrix. Prove that the product of the perpendiculars on it from the extremities of any chord passing through that focus is constant.

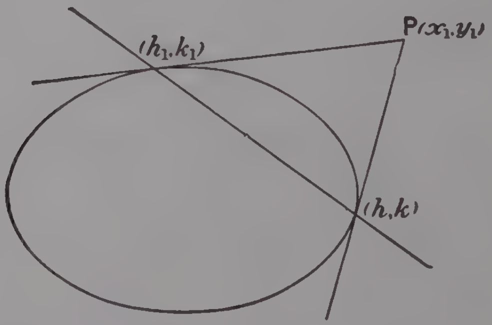
23. Prove that on the ellipse half of the minor-axis is a mean proportional between **AF** and FA_1.

24. In a diagram mark two points **A**, A_1 on the axis of x distant $1\frac{1}{2}$ inches from the origin, and two points, **B**, B_1 on the axis of y distant $\frac{3}{4}$ inch from the origin. Draw a line PP_1 making intercepts $3\frac{3}{16}$ and $\frac{17}{20}$ inches on the axes. If PP_1 is a tangent to the ellipse whose axes are the lines AA_1 and BB_1, find the foci and the point of contact of the tangent.

POLE AND POLAR

137. To find the equation of the chord of contact of tangents drawn from a point to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Let (x_1, y_1) be the co-ordinates of the point from which the tangents are drawn. Let (h, k) and (h_1, k_1) be the co-ordinates of the points of contact of the tangents.

The equations of the tangents at (h, k) and (h_1, k_1) are

$$\frac{xh}{a^2} + \frac{yk}{b^2} = 1.$$

and $$\frac{xh_1}{a^2} + \frac{yk_1}{b^2} = 1.$$

Since the point (x_1, y_1) is on both of these lines

$$\frac{x_1h_1}{a^2} + \frac{y_1k}{b^2} = 1. \tag{1}$$

and $$\frac{x_1h_1}{a^2} + \frac{y_1k_1}{b^2} = 1. \tag{2}$$

But equations (*1*) and (*2*) are the conditions that the points (h, k) and (h_1, k_1) may lie on the straight line whose equation is $\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1$ which is the required equation of the chord of contact of the tangents drawn from the point (x_1, y_1).

Compare this equation with the equation of a tangent at (x_1, y_1). *See* § *111.*

The equation of the polar of **P** (x_1, y_1) with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ may be deduced as in § *74* and § *112* and is of the form $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

In the ellipse, if the polar of **P** passes through **Q**, the polar of **Q** passes through **P**.

This may be proved as in § *115.*

138. To find the pole of the straight line $\mathrm{Ax} + \mathrm{By} + \mathrm{C} = 0$ with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Let (x_1, y_1) be the co-ordinates of the pole.

The equation of the polar of (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \tag{1}$$

This equation must be the same as

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0 \tag{2}$$

Comparing (*1*) and (*2*)

$$\frac{\frac{x_1}{a^2}}{\mathrm{A}} = \frac{\frac{y_1}{b^2}}{\mathrm{B}} = \frac{-1}{\mathrm{C}}$$

Therefore $x_1 = \frac{-\mathrm{A}}{\mathrm{C}} \cdot a^2$, $y_1 = \frac{-\mathrm{B}}{\mathrm{C}} \cdot b^2$.

Therefore the pole is $\left(-\frac{\mathrm{A}}{\mathrm{C}} \cdot a^2, -\frac{\mathrm{B}}{\mathrm{C}} \cdot b^2\right)$.

139.—Exercises

1. Find the polar of the following points:

   (*a*) (3, 5) with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

   (*b*) (4, 8) with respect to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

   (*c*) (3, 7) with respect to the ellipse $4x^2 + 9y^2 = 36$

2. Find the pole of the straight line:

   (*a*) $2x + 3y = 1$ with respect to ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

   (*b*) $3x + 2y = 4$ with respect to ellipse $2x^2 + y^2 = 12$.

   (*c*) $ax + y = c$ with respect to ellipse $4x^2 + 9y^2 = 36$.

3. Find the pole of the line $\frac{x}{a} + \frac{y}{b} = 1$ with respect to the ellipse $\frac{x^2}{4} + \frac{y^2}{25} = 1$.

4. Find the pole of the line $lx + my + n = 0$ with respect to the ellipse $4x^2 + y^2 = 16$.

5. Find the pole of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1).

6. Find the equation of the chord of contact of tangents to $x^2 + 9y^2 = 25$ at the point $(4, 7)$.

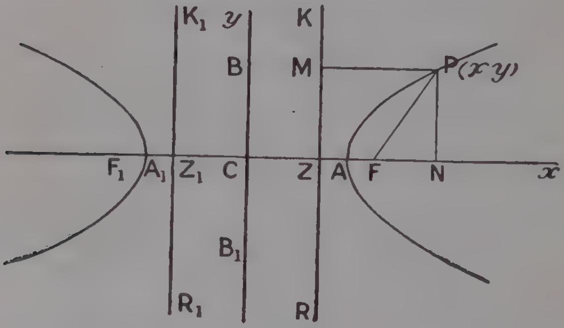
7. If the focal chord of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ makes an angle of 60° with the x-axis, find its pole.

CHAPTER VI

THE HYPERBOLA

140. **An Hyperbola** is the locus of a point which moves so that its distance from a fixed point, called the focus is in a constant ratio, which is greater than unity, to its distance from a fixed straight line called the directrix.

To find the equation of the hyperbola.



Let F be the focus and KR the directrix; draw FZ perpendicular to KR. Divide FZ internally at A and externally at A_1

so that $\frac{AF}{ZA} = e$ or $AF = e.ZA$ (*1*)

and $\frac{A_1F}{A_1Z} = e$ or $A_1F = e.A_1Z$ (*2*)

By definition A and A_1 are points on the locus.

Bisect A_1A at C; let $A_1A = 2a$.

Adding (*1*) and (*2*)

$$AF + A_1F = e\,(ZA + A_1Z)$$

$$\therefore\ 2CF = e.2a$$

$$\text{or } CF = ae \qquad (3)$$

Subtracting (*1*) from (*2*)

$$A_1F - AF = e\,(A_1Z - ZA)$$

$$\therefore\ A_1A = e\,(A_1A - 2ZA)$$

$$\therefore\ 2CA = e.2CZ$$

$$\text{or } CZ = \frac{a}{e} \qquad (4)$$

Let **C** be taken as origin, **CZ** as the axis of x and **CY**, which is perpendicular to **CZ**, as the axis of y. Let **P** (x,y) be any point on the locus. Draw **PN** perpendicular to **CZ** and **PM** perpendicular to **KR**. Then **CN** $= x$, **PN** $= y$; the co-ordinates of **F** and **Z** are (ae,o) and $\left(\frac{a}{e},o\right)$

By definition of the hyperbola

$$\frac{PF}{MP} = e$$

$$\text{or } PF^2 = e^2.MP^2 = e^2.ZN^2.$$

$$\therefore\quad FN^2 + PN^2 = e^2 \cdot ZN^2$$

But **FN** $=$ **CN** $-$ **CF** and **ZN** $=$ **CN** $-$ **CZ**

$$\therefore\ (CN - CF)^2 + PN^2 = e^2\,(CN - CZ)^2$$

$$\therefore\ (x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2$$

Whence $x^2\,(e^2 - 1) - y^2 = a^2\,(e^2 - 1)$

$$\text{or } \frac{x^2}{a^2} - \frac{y^2}{a^2\,(e^2 - 1)} = 1. \qquad (5)$$

Which is the equation of the hyperbola.

Note: the
form $x^2(1-e$
ca [illegible]
used
b² with [illegible]

Putting $a^2 (e^2 - 1) = b^2$, equation (*5*) takes the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is the form in which the equation is usually written.

Equation (*3*) gives the distance from the centre to the focus, $\mathsf{CF} = ae$.

Equation (*4*) gives the distance from the centre to the directrix, $\mathsf{CZ} = \frac{a}{e}$

The square of the eccentricity, $e^2 = \frac{a^2 + b^2}{a^2}$ is obtained from $a^2 (e^2 - 1) = b^2$

The points A and A_1 are called the **Vertices** of the hyperbola, C is the **Centre**, A_1A is the **Transverse** axis of the curve, BB_1 is called the **Conjugate** axis where B and B_1 are two points on the axis of Y such that $\mathsf{CB} = b$, $\mathsf{CB}_1 = -b$. Any chord through the centre is called a **Diameter.**

141. In deducing the equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the focus and directrix were placed in convenient positions in order to get the equation in the most suitable form from which to study the properties of the hyperbola. In the following examples the focus is taken as any point and the directrix as any straight line.

To find the equation of an hyperbola whose focus, directrix and eccentricity are given.

Let the focus be the point (4, 5), the directrix, the line $2x + 3y + 4 = 0$ and the eccentricity $\frac{7}{6}$.

Then by definition, if (x, y) is any point on the locus,

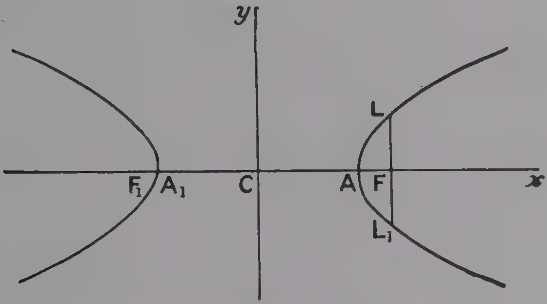
$$(x-4)^2+(y-5)^2=\left(\frac{7}{6}\right)^2\frac{(2x+3y+4)^2}{13}$$

The equation of the hyperbola whose focus is the point (h,k), whose directrix is the line $\mathsf{A}x+\mathsf{B}y+\mathsf{C}=0$ and whose eccentricity is e

$$\text{is } (x-h)^2+(y-k)^2=e^2\frac{(\mathsf{A}x+\mathsf{B}y+\mathsf{C})^2}{\mathsf{A}^2+\mathsf{B}^2}$$

Unless otherwise stated, the equation of the hyperbola will be assumed to be of the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

142. To find the length of a latus rectum of the hyperbola.



$\mathsf{LFL_1}$ is a latus rectum of the hyperbola.

Substitute $x = ae$ in the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Then $\frac{a^2e^2}{a^2}-\frac{y^2}{b^2}=1$

or $\frac{y^2}{b^2}=e^2-1$

$$= \frac{a^2 + b^2}{a^2} - 1$$

$$= \frac{b^2}{a^2}$$

$$\therefore \quad y^2 = \frac{b^4}{a^2} \text{ or } y = \frac{b^2}{a}$$

$\therefore$ the length of latus rectum $= \dfrac{2b^2}{a}$.

143. The following example illustrates how an equation may be transformed so as to give important features of the curve.

The equation $16x^2 - 9y^2 + 32x + 54y - 209 = 0$ may be written $(16x^2+32x+16) - (9y^2-54y+81) = 144$

or $16\,(x + 1)^2 - 9\,(y - 3)^2 = 144$

or $\dfrac{(x + 1)^2}{9} - \dfrac{(y - 3)^2}{16} = 1$

Hence

(*1*) The centre is at the point $(-1, 3)$

(*2*) The transverse axis is parallel to the x axis.

(*3*) $a = 3$, $b = 4$.

(*4*) The vertices are $(2, 3)$ and $(-4, 3)$

(*5*) The eccentricity, $e = \sqrt{\dfrac{9 + 16}{9}} = \dfrac{5}{3}$

(*6*) The foci are at $(4, 3)$ and $(-6, 3)$

(*7*) The directrices are $x = \dfrac{4}{5}$, $x = -\dfrac{14}{5}$.

144.—Exercises

1. Plot the locus of the following equations:

(a) $\frac{x^2}{4} - \frac{y^2}{9} = 1$

(b) $\frac{x^2}{9} - \frac{y^2}{4} = 1$

(c) $\frac{x^2}{5} - \frac{y^2}{12} = 8$

(d) $\frac{x^2}{4} - \frac{y^2}{9} = -1$

2. On the same piece of squared paper plot the following equations and points:

(a) $\frac{x^2}{4} - \frac{y^2}{25} = 1$

(b) $(\sqrt{29}, 0)$

(c) $(-\sqrt{29}, 0)$

(d) $x = \frac{4}{\sqrt{29}}$

(e) $x = \frac{-4}{\sqrt{29}}$

3. Find the eccentricity, the co-ordinates of the foci and length of the latus rectum in the following hyperbolas:

(a) $3x^2 - 4y^2 = 12$

(b) $5x^2 - y^2 = 9$

4. Determine the lengths of axes and latus rectum, co-ordinates of vertices and foci and equations of directrices for the following hyperbolas:

(a) $9y^2 - 4x^2 + 36 = 0.$

(b) $\frac{x^2}{16} - \frac{y^2}{12} = 1.$

5. Find the equations of the following hyperbolas whose centres are at (o, o) and whose transverse axes are along the x-axis.

(a) Transverse axis = 4, conjugate axis equals one-half the the distance between the foci.

(*b*) Transverse axis $= 6$, and $(6, 3)$ is the point on the curve.

(*c*) Transverse axis $= 8$ and focus is at $(6, o)$.

(*d*) Latus rectum $= 3$ and $a = 2b$.

(*e*) $e = 2$ and distance from focus to nearest vertex $= 1$.

6. Find the length of the latus rectum of the following hyperbolas:

(*a*) $\frac{x^2}{4} - \frac{y^2}{16} = 1$

(*b*) $4x^2 - 16y^2 = 1$.

7. Find the equations of the hyperbolas determined by the following data:

(*a*) $a = 4, b = 3$.

(*b*) $a = 6, e = 2$.

(*c*) $b = 3, e = \frac{4}{3}$.

8. Find the equation of the following hyperbola, deduce the coordinates of its foci, its eccentricity and the length of its latus rectum: centre $(-1, 3)$ transverse axis is parallel to the x-axis and equal 8, conjugate axis $=10$.

9. Find the equation of an hyperbola whose eccentricity is 2 and having one focus at the point $(-1, -2)$ and the corresponding directrix $y = 4$.

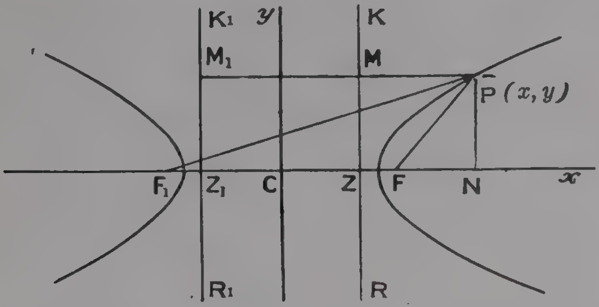
10. In an hyperbola whose centre is the origin and whose transverse axis is the axis of x the distance from the centre to a focus is 18 and to the directrix is 8; find the equation of the curve.

11. Find the equation of the hyperbola whose eccentricity is $\frac{3}{2}$, directrix $4x + 6y - 15 = 0$ and focus $(3, 4)$.

12. Prove that the latus rectum of an hyperbola is a third proportional to the transverse and conjugate axis.

13. Find the point at which the straight line $25x + 12y - 45 = 0$ cuts the hyperbola $25x^2 - 9y^2 = 225$.

145. The difference of the focal distances of any point on an hyperbola is constant and equal to $2a$.



Let $P\ (x, y)$ be any point on the hyperbola, F and F_1 the foci, KR and K_1R_1 the directrices.

It is required to prove $PF_1 - PF = 2a$.

$$PF = e \cdot MP = e \cdot ZN$$

$$= e\ (CN - CZ)$$

$$= e\left(x - \frac{a}{e}\right) = ex - a$$

$$PF_1 = e \cdot M_1P = e \cdot Z_1N$$

$$= e\ (CN + Z_1C)$$

$$= e\left(x + \frac{a}{e}\right) = ex + a$$

Therefore $PF_1 - PF = (ex + a) - (ex - a)$

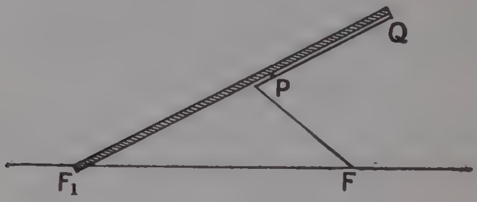
$$= 2a.$$

This property leads to another definition of an hyperbola.

An hyperbola is the locus of a point which moves so that the difference of its distances from two fixed points is constant.

146. Mechanical method for constructing an hyperbola.

The property of an hyperbola deduced in the last article provides a simple method for constructing an hyperbola mechanically if the foci and length of the transverse axis are given.



Let **F** and F_1 be the foci, $2a$ the transverse axis.

Obtain an inextensible string of length l and a ruler whose length is greater than the transverse axis by l, *i.e.* the length of the ruler is $2a + l$. Let the ruler, F_1Q, be placed so that it can rotate about F_1; fasten one end of the string at **Q** and the other end at **F**.

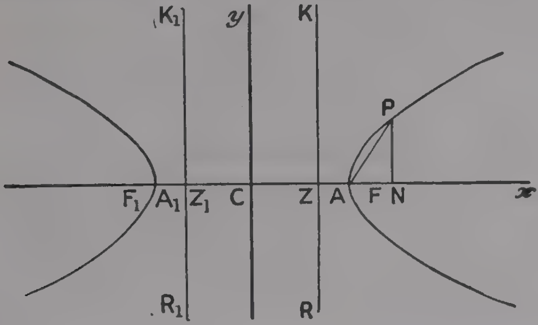
Let the ruler F_1Q rotate about F_1 and keep the string taut by a pencil at **P**.

The curve traced out is an hyperbola.

$$\begin{aligned}\text{Because } PF_1 - PF &= (QF_1 - QP) - (l - QP)\\ &= 2a + l - QP - l + QP\\ &= 2a\end{aligned}$$

Hence the locus of **P** follows the law of the hyperbola.

147. To trace the form of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from its equation.



(*1*) If $x = 0$, $y = \pm b\sqrt{-1}$, which is imaginary. Hence the y-axis does not cut the curve.

(*2*) If $y = 0$, $x = \pm a$. Hence the curve cuts the x-axis at distances a, $-a$, from the origin.

(*3*) $y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$. Hence x cannot be numerically less than $\pm a$. Therefore no part of the curve lies between $x = -a$ and $x = a$.

(*4*) $y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$. Hence as x increases numerically beyond $\pm a$, y also increases and when x becomes indefinitely great, y also becomes indefinitely great. Therefore the curve has infinite branches on both sides of the origin, both above and below the x-axis.

Also, for a given value of x, the values of y are equal with opposite signs. Hence the curve is symmetrical with respect to the x-axis. Similarly, since $x = \pm \frac{a}{b}\sqrt{y^2 + b^2}$ the curve is symmetrical with respect to the y-axis.

(5) If the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be cut by the line $y = mx + c$ the abscissae of the points of intersection are given by the equation.

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

Since this is a quadratic equation, x has two values. Therefore a straight line can cut an hyperbola in only two points, which are real, coincident or imaginary.

(6) If P (x, y) be any point on the curve,

tan PAN

$$= \frac{\mathrm{PN}}{\mathrm{AN}} = \frac{y}{x-a} = \frac{y^2}{(x-a)y} = \frac{b^2}{a^2}\frac{(x^2-a^2)}{(x-a)y} = \frac{b^2}{a^2}\cdot\frac{x+a}{y}$$

Ultimately when P moves along the curve indefinitely close to A the value of $y = 0$ and $x = a$.

Then *tan* PAN $= \frac{b^2}{a^2} \cdot \frac{a+a}{0} = \infty$

Therefore angle PAN in the limit = 90°.

Hence the curve cuts the axis of x at right angles at the point A.

From these facts it is evident that the hyperbola has the form given in the diagram.

The symmetry of these relations shows that since there is a focus F and a directrix KR to the right of the origin there is also a focus F_1 and a directrix $\mathrm{K}_1\mathrm{R}_1$ at the same distances to the left of the origin.

148. Every chord through the centre of the hyperbola is there bisected.

Let (x_1, y_1) be any point on the curve. The co-ordinates (x_1, y_1) will satisfy the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\therefore \frac{x_1{}^2}{a^2} - \frac{y_1{}^2}{b^2} = 1$$

$$\text{also } \frac{(-x_1)^2}{a^2} - \frac{(-y_1)^2}{b^2} = 1$$

i.e. the point $(-x_1, -y_1)$ is also on the curve.

But the points (x_1, y_1) and $(-x_1, -y_1)$ are on a straight line $\frac{x}{x_1} - \frac{y}{y_1} = 0$ which passes through the origin and are equidistant from the origin since

$$\sqrt{x_1{}^2 + y_1{}^2} = \sqrt{(-x_1)^2 + (-y_1)^2}.$$

Hence every diameter is bisected at the centre of the hyperbola.

149.—Exercises

1. Prove that the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and the hyperbola $\frac{x^2}{9} - \frac{y^2}{7} = 1$ have the same centre and foci.

2. Find the locus of a point which moves so that its distance from the origin is a mean proportional between its distances from the points $(3, 0)$, $(-3, 0)$.

3. Find the equation of the locus of the point which moves so that the difference of its distances from two points $(6, 0)$ and $(-6, 0)$ is equal to 10.

4. Find the equation of the hyperbola, referred to its axes as axes of co-ordinates,

   (*a*) whose transverse and conjugate axes are respectively 6 and 8.

   (*b*) whose conjugate axis is 5 and the distance between whose foci is 13.

(*c*) whose conjugate axis is 7 and which passes through the point $(3, -2)$

(*d*) The distance between whose foci is 16 and whose eccentricity is $\sqrt{2}$.

5. Find the equation of an hyperbola whose foci are $(8, 0)$ and $(-8, 0)$ and whose conjugate axis equals 6.

6. If the transverse axis equals $2a$ find the equation of the hyperbola whose axes are axes of co-ordinates and whose vertex bisects the distance between the centre and the focus.

7. Prove that the straight lines $\frac{x}{a} - \frac{y}{b} = K$ and $\frac{x}{a} + \frac{y}{b} = \frac{1}{k}$ always meet on an hyperbola.

8. A point (x, y) moves so that the difference of its distances from $(10, 10)$ and $(-10, -10)$ is always 20. Find the equation of its path.

9. A point moves in such a way that its distance from $(25, 0)$ is always $\frac{5}{4}$ of its distance from the line $x = 16$. Find the equation of the path.

10. In an hyperbola a line from the centre to an extremity of a latus rectum makes an angle of 45° with the transverse axis. Find the eccentricity.

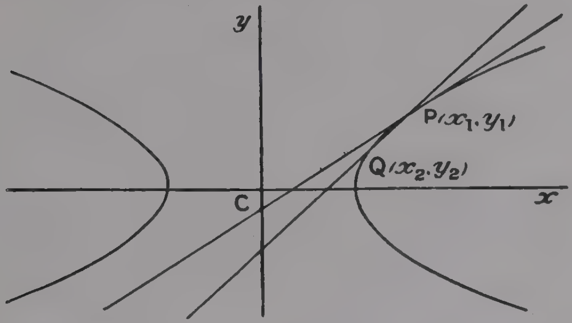
11. In an hyperbola, a diameter whose length is 4 make an angle of 30° with the transverse axis. If the eccentricity is $\sqrt{3}$, find the equation of the hyperbola.

12. Prove that the point $(4, 1)$ is on the hyperbola $\frac{x^2}{8} - \frac{y^2}{1} = 1$ and that the difference of its focal distances is equal to the transverse axis.

13. If the crack of a rifle and the thud of the ball on the target are heard at the same instant, prove that the locus of the hearer is an hyperbola for which the rifle is the farther and the target the nearer focus.

TANGENTS AND NORMALS

150. To find the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of the co-ordinates of the point of contact P (x_1, y_1).



Let Q (x_2, y_2) be another point on the hyperbola.

Then the equation of PQ is $\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$ (*1*)

Since P and Q are both on the hyperbola

$$\text{then } \frac{x^2_1}{a^2} - \frac{y^2_1}{b^2} = 1$$

$$\text{and } \frac{x^2_2}{a^2} - \frac{y^2_2}{b^2} = 1.$$

Subtracting $\frac{1}{a^2}(x^2_1 - x^2_2) - \frac{1}{b^2}(y^2_1 - y^2_2) = 0.$

$$\text{or } \frac{1}{a^2}(x_1 - x_2)(x_1 + x_2) = \frac{1}{b^2}(y_1 - y_2)(y_1 + y_2) \qquad (2)$$

Multiplying the terms of (*2*) by the equal fractions in (*1*)

$$\frac{(x_1 + x_2)(x - x_1)}{a^2} = \frac{(y_1 + y_2)(y - y_1)}{b^2} \qquad (3)$$

If now **PQ** rotates about **P** until **Q** coincides with **P** then $x_2 = x_1$ and $y_2 = y_1$

and equation (*3*) becomes

$$\frac{2x_1(x - x_1)}{a^2} = \frac{2y_1(y - y_1)}{b^2}$$

$$\text{or } \frac{xx_1}{a^2} - \frac{x_1^2}{a^2} = \frac{yy_1}{b^2} - \frac{y_1^2}{b^2}$$

$$\text{or } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

$\therefore \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ is the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) (*4*)

Cor. 1.—The co-ordinates at the extremities of the transverse axis are (a, o) and $(-a, o)$ respectively. Substituting these co-ordinates in (*4*) the tangents at these points are $x = a$ and $x = -a$.

Hence the tangents at the extremities of the transverse axis are perpendicular to that axis and are parallel to the conjugate axis.

151. To find the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in terms of m, the slope of the tangent.

The tangent may be represented by the line $y = mx + c$ where c is yet to be found.

To find the abscissae of the points where the line $y = mx + c$ cuts the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, eliminate y

by substitution, giving

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1$$

$$\text{or } x^2 (b^2 - a^2m^2) - 2mcxa^2 - a^2 (c^2 + b^2) = 0$$

$$\text{or } x^2 (m^2a^2 - b^2) + 2mcxa^2 + a^2 (c^2 + b^2) = 0$$

If the line is a tangent, the values of x from this equation are equal to each other.

The condition for equal values of x is

$$m^2c^2a^4 = (m^2a^2 - b^2)(c^2 + b^2) a^2$$

$$\text{or } c = \pm\sqrt{m^2a^2 - b^2}.$$

Thus the equation of the tangent is

$$y = mx \pm \sqrt{m^2a^2 - b^2}.$$

Alternative proof of the preceding proposition:

(*1*) It has been shown that $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ is the tangent at the point (x_1, y_1).

Comparing $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$

with $y = mx + c$

or $mx - y = -c$

It gives $\dfrac{m}{\frac{x_1}{a^2}} = \dfrac{1}{\frac{y_1}{b^2}} = \dfrac{-c}{1}$

$$\text{Hence } c = \frac{-ma}{\frac{x_1}{a}} = \frac{-b}{\frac{y_1}{b}} = \pm\frac{\sqrt{m^2a^2 - b^2}}{\sqrt{\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}}} = \pm\frac{\sqrt{m^2a^2 - b^2}}{1}$$

$\therefore$ equation becomes $y = mx \pm \sqrt{m^2a^2 - b^2}$.

152. To find the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1).

The equation of any line through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1) \tag{1}$$

If this be the normal at (x_1, y_1) it is perpendicular to the tangent

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

The condition for perpendicularity is

$$m \cdot \frac{x_1}{a^2} + \frac{y_1}{b^2} = 0.$$

$$\text{or } m = -\frac{y_1}{b^2} \cdot \frac{a^2}{x_1}$$

Substituting this value of m in *(1)* thus making *(1)* the normal, the equation is

$$y - y_1 = -\frac{y_1}{b^2} \cdot \frac{a^2}{x_1}(x - x_1)$$

$$\text{or } \frac{b^2}{y_1}(y - y_1) = -\frac{a^2}{x_1}(x - x_1)$$

$$\text{or } \frac{a^2}{x_1} \cdot x + \frac{b^2}{y_1} y = a^2 + b^2.$$

This equation is frequently written in the form

$$\frac{x - x_1}{\frac{x_1}{a^2}} = -\frac{y - y_1.}{\frac{y_1}{b^2}}$$

153.—Exercises

1. Write the equations of the tangent and normal to each of the following hyperbolas at the point indicated:

(*a*) $\frac{x^2}{4} - \frac{y^2}{9} = 1$ at point 4, $3\sqrt{3}$.

(*b*) $2x^2 - 4y^2 = 12$ at point 3, $\frac{\sqrt{3}}{\sqrt{2}}$.

(*c*) $4x^2 - 6y^2 = 1$ at point $-2, \frac{\sqrt{5}}{\sqrt{2}}$.

2. Find equation of tangents to hyperbola $\frac{x^2}{9} - \frac{y^2}{1} = 1$.

making (*1*) angle of 30° with x-axis.
(*2*) angle of 45° with x-axis.

3. Find the equations of tangents to the hyperbola $3x^2 - 4y^2 = 12$ at the points whose ordinates are $+4$.

4. Find the equations of the tangent and normal to the following hyperbolas at the points indicated:

(*a*) $4x^2 - y^2 = 6$, abscissae = 2, ordinate negative.
(*b*) $6x^2 - 5y^2 = 20$ at positive extremity of latus rectum.

5. Find the values of m that the line $y = mx$ may be a tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

6. For what values of m will the line $y = mx$ be a tangent to the hyperbola $4x^2 - 9y^2 = 1$.

7. Find the relation holding between A and B if $Ax + By = 0$ is a tangent to the hyperbola $2x^2 - 5y^2 = 10$.

8. Find the equation of the normal to the hyperbola $\frac{x^2}{4} - \frac{y^2}{6} = 1$ at the point 4, $3\sqrt{2}$.

9. Find the equations of the normals to the hyperbola $4x^2 - 9y^2 = 12$ drawn at the extremities of the latus rectum.

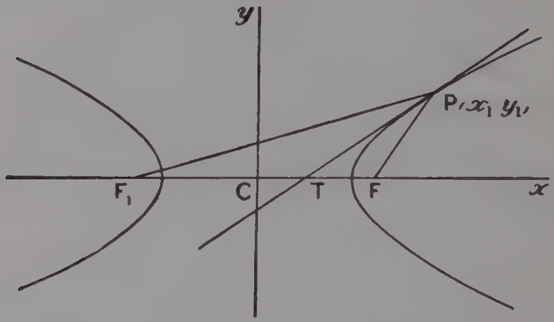
10. Find the points of intersection of the line $2x + 3y = 7$ and the hyperbola $2x^2 - 3y^2 = 5$.

11. Find the equations of the normals to the hyperbola $2x^2 - 3y^2 = 5$ at the points of intersection of the line $2x + 3y = 7$ and the hyperbola.

12. Find the equations of the tangents to the hyperbola $x^2 - 4y^2 = 36$ perpendicular to $6x - 4y + 9 = 0$.

13. Find the angle between the tangents drawn at the extremities of a latus rectum of the hyperbola $9x^2 - 16y^2 = 144$.

154. At any point on the hyperbola the tangent bisects the angle between the focal distances.



Let **P** (x_1, y_1) be any point on the hyperbola; **PT** the tangent; **PF**, **PF**$_1$ the focal distances.

It is required to prove angles **FPT**, **F**$_1$**PT** are equal.

Equation of tangent **PT** is $\dfrac{xx_1}{a^2} - \dfrac{yy_1}{b^2} = 1$. (*1*)

At the point **T**, $x = \mathbf{CT}$, $y = 0$.

Substituting these values in (*1*)

$$\frac{x_1 \cdot \mathbf{CT}}{a^2} = 1$$

$$\text{or } \mathbf{CT} = \frac{a^2}{x_1}$$

$$\therefore \quad F_1T = ae + \frac{a^2}{x_1} = \frac{a}{x_1}(ex_1 + a)$$

$$\text{and } TF = ae - \frac{a^2}{x_1} = \frac{a}{x_1}(ex_1 - a)$$

$$\text{Hence} \quad \frac{F_1T}{TF} = \frac{ex_1 + a}{ex_1 - a}$$

But $F_1P = ex_1 + a$ and $FP = ex_1 - a$.

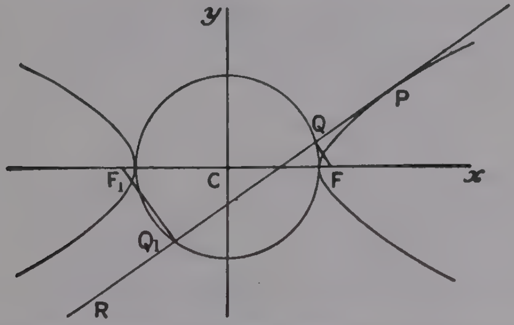
$$\therefore \quad \frac{F_1T}{TF} = \frac{F_1P}{FP}$$

Hence **PT** bisects the angle **F₁PF.**

Cor.—This proposition provides a short method for drawing a tangent to the hyperbola at any point **P**.

Having constructed the hyperbola, join the foci, **F** and **F₁** to the point **P**; then bisect the angle **F₁PF** by **PT**. The line **PT** is the tangent at **P**.

155. In the hyperbola the locus of the foot of the perpendicular dropped from either focus on the tangent is a circle on the transverse axis as diameter.



Let **PR** be a tangent at any point **P**, **FQ**, **F₁Q₁** perpendiculars from the foci on **PR**.

Let equation of **PR** be

$$y = mx + \sqrt{m^2a^2 - b^2} \tag{1}$$

The line **FQ** through the focus (ae, o) and perpendicular to **PR** is
$$y = -\frac{1}{m}(x - ae) \tag{2}$$

From (*1*)
$$y - mx = \sqrt{m^2a^2 - b^2} \tag{3}$$

From (*2*)
$$my + x = ae = \sqrt{a^2 + b^2}. \tag{4}$$

If we eliminate m from (*3*) and (*4*) the resulting equation is the locus of the foot of perpendiculars from the focus (ae, o) on any tangent to the hyperbola.

Squaring (*3*), $y^2 - 2m\,xy + m^2x^2 = m^2a^2 - b^2$.

Squaring (*4*), $m^2y^2 + 2m\,xy + x^2 = a^2 + b^2$.

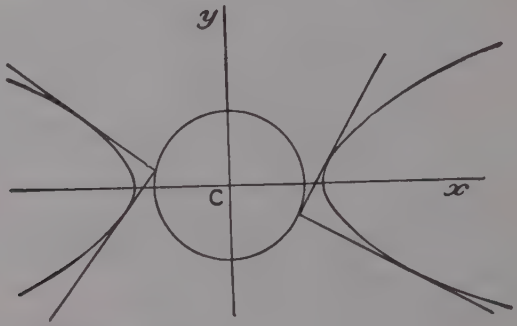
Adding $y^2(1 + m^2) + x^2(1 + m^2) = a^2(1 + m^2)$.

or $x^2 + y^2 = a^2$ which is the equation of a circle on the transverse axis as diameter.

The same equation is obtained if the line **F₁Q₁** through the focus $(-ae, o)$ is considered.

This circle is called the **Auxiliary Circle** of the hyperbola.

156. **In the hyperbola the locus of the intersection of tangents at right angles to each other is the circle** $x^2 + y^2 = a^2 - b^2$.



The line whose equation is

$$y = mx + \sqrt{m^2a^2 - b^2} \qquad (1)$$

is a tangent to the hyperbola for all values of m.

This equation gives us the directions of the tangents which pass through the point (x, y).

Rationalizing (*1*) it becomes

$$m^2 (x^2 - a^2) - 2mxy + y^2 + b^2 = 0. \qquad (2)$$

Let m_1 and m_2 be the roots of (*2*)

If the tangents are at right angles, $m_1m_2 = -1$.

The product of the roots of (*2*) is $\dfrac{y^2 + b^2}{x^2 - a^2}$

Therefore $\dfrac{y^2 + b^2}{x^2 - a^2} = -1$

or $x^2 + y^2 = a^2 - b^2$

which is the equation of a circle.

Therefore the required locus is a circle.

This circle is called the **Director Circle** of the hyperbola.

EQUILATERAL OR RECTANGULAR HYPERBOLA

157.—An equilateral or rectangular hyperbola is one in which the lengths of the transverse and conjugate axes are equal.

In $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, let $b = a$, then the equation of the equilateral hyperbola becomes $x^2 - y^2 = a^2$.

The eccentricity of the equilateral hyperbola is $\sqrt{2}$

$$\text{for } e^2 = \frac{a^2 + b^2}{a^2}$$

$$= \frac{2a^2}{a^2}$$

$$\therefore \quad e = \sqrt{2}$$

158.—Exercises

1. Find the equation of an equilateral hyperbola which passes through $(3, -1)$ and has its axes on the co-ordinate axes.

2. Show (*a*) that **P** $(6, 2\sqrt{3})$ is on the hyperbola $4x^2 - 9y^2 = 36$.
   (*b*) Find the equation of tangent at point **P**.
   (*c*) Find the equation of line bisecting the external angles between the lines joining **P** to the foci.

3. **PT** is a tangent to the hyperbola $4x^2 - y^2 = 13$ at the point $(2, \sqrt{3})$. Find the angle between **PT** and **PF** if **F** is the focus and **T** is the point where the tangent meets the x-axis.

4. In an equilateral hyperbola prove the angle subtended by any chord at the centre is the supplement of the angle between the tangents at the ends of the chord.

5. In and equilateral hyperbola, show that if the normal at **P** meet the axes in **G** and g then **PG** = **P**g = **PC**.

6. **P** is any point on a hyperbola whose vertex is **V** and centre **C**. The tangent at **P** meets **CV** at **M** and the tangent at **V** meets **CP** at **S**. Prove **MS** and **VP** are parallel.

7. Perpendiculars are dropped from the foci of the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ to a tangent at any point on the hyperbola.

Find (*1*) the length of each perpendicular.

(*2*) their product.

8. Find the locus of the feet perpendiculars dropped from the foci on any tangent to the hyperbola $3x^2 - y^2 = 10$.

9. A series of tangents are drawn to the hyperbola $x^2 - 4y^2 = 12$. Find the locus of the intersection of the tangents that are at right angles and show that it is a circle whose centre is the centre of the hyperbola.

10. Find the equation of the director-circle of the hyperbola $4x^2 - 9y^2 = 12$.

11. Find the equations of the tangents to the director-circle of the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ which are

(*1*) parallel to x-axis.

(*2*) perpendicular to x-axis.

12. Find the equation of the chord of the hyperbola $x^2 - 2y^2 = 4$ through (3, 1) which is bisected by the diameter $2y = x$.

13. Find the equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$ which is bisected at point (5, 3).

14. Find the equation of the common tangent to the two hyperbolas:

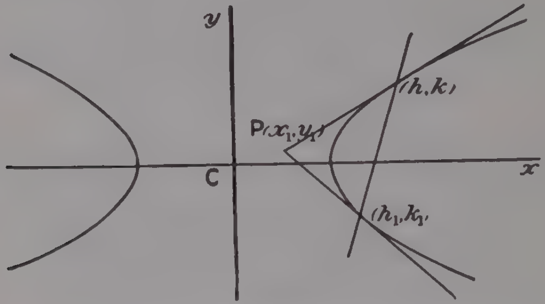
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

15. If two tangents be drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from any point show that they are in the same ratio as the corresponding normals.

POLE AND POLAR

159. To find the equation of the chord of contact of tangents drawn from an outside point to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.



Let (x_1, y_1) be the co-ordinates of the point from which the tangents are drawn. Let (h, k) and (h_1, k_1) be the co-ordinates of the points of contact of the tangents.

The equations of the tangents at (h, k) and (h_1, k_1) are

$$\frac{xh}{a^2} - \frac{yk}{b^2} = 1$$

$$\text{and } \frac{xh_1}{a^2} - \frac{yk_1}{b^2} = 1$$

Since the point (x_1, y_1) is on both of these lines

$$\text{then } \frac{x_1h}{a^2} - \frac{y_1k}{b^2} = 1 \qquad (1)$$

$$\text{and } \frac{x_1h_1}{a^2} - \frac{y_1k_1}{b^2} = 1 \qquad (2)$$

But equations (*1*) and (*2*) are the conditions that the points (h, k) and (h_1, k_1) may be on the straight line whose equation is

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1$$

which is the required equation of the chord of contact of the tangents drawn from the point (x_1, y_1)

Compare this equation with the equation of the tangent at (x_1, y_1). See § 111.

The equation of the polar of P (x_1, y_1) with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ may be deduced as in § 74 and § 112 and is of the form $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

In the hyperbola if the polar of P passes through Q, the polar of Q passes through P.

This may be proved as in § 115.

Cor.—If the point Q moves along the polar of P, the polar of Q changes its position but always passes through P.

Therefore if the pole moves along a straight line, its polar turns about the pole of that line.

160. To find the pole of the straight line Ax + By + C = 0 with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Let (x_1, y_1) be the co-ordinates of the pole. The equation of the polar of (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \qquad (1)$$

This equation must be the same as

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C} = 0 \tag{2}$$

Comparing (*1*) and (*2*)

$$\frac{\frac{x_1}{a^2}}{\mathrm{A}} = \frac{\frac{-y_1}{b^2}}{\mathrm{B}} = \frac{-1}{\mathrm{C}}$$

$$\therefore\ x_1 = \frac{-\mathrm{A}}{\mathrm{C}}\cdot a^2,\ y_1 = \frac{\mathrm{B}}{\mathrm{C}}\cdot b^2$$

$$\therefore\ \text{The pole is } \left(-\frac{\mathrm{A}}{\mathrm{C}}\cdot a^2, \frac{\mathrm{B}}{\mathrm{C}}\cdot b^2\right).$$

161.—Exercises

1. Find the polar of the point

(*a*) (3, 5) with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(*b*) (1, 6) with respect to the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$

(*c*) (2, 5) with respect to the hyperbola $4x^2 - 9y^2 = 36$.

2. Find the pole of the straight line

(*a*) $2x + 3y = -5$ with respect to $\frac{x^2}{2} - \frac{y^2}{3} = 1$

(*b*) $3x + y = 6$ with respect to $2x^2 - 3y^2 = 12$

(*c*) $ax + y = c$ with respect to $4x^2 - 9y^2 = 36$.

3. Find the pole of the line $\frac{x}{a} + \frac{y}{b} = 1$ with respect to the hyperbola $\frac{x^2}{4} - \frac{y^2}{25} = 1$.

4. Find the pole of the line $1x + my + n = 0$ with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

5. Find the pole of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1).

6. Find the equation of the chord of contact of tangents to $4x^2 - 9y^2 = 20$ from the point $(1, 6)$.

7. If a focal chord of the hyperbola $\frac{x^2}{9} - \frac{y^2}{7} = 1$ makes an angle of 45° with x-axis, find its pole.

8. Tangents are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from any point on one of the directrices. Prove that the line drawn through the points of tangency passes through the corresponding focus.

9. An ellipse and an hyperbola have the same principal axes. Show that the polar of any point on either curve with respect to the other touches the first curve.

10. In an hyperbola show that the polar of any point is parallel to the chord which is bisected at that point.

11. If Q and P be any two points; and C the centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; show that the perpendiculars from Q and C on the polar of P with respect to the hyperbola are to one another in the same ratio as the perpendiculars from P and C on the polar of Q.

ASYMPTOTES

162. An **Asymptote** to any conic is the limiting position of the tangent as the point of contact moves off to an infinite distance, the tangent itself remaining at a finite distance from the origin.

163. To find the asymptotes of the hyperbola.

The tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1)

$$\text{is } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

$$\text{or } \frac{x}{a^2}\cdot\frac{x_1}{y_1} - \frac{y}{b^2} = \frac{1}{y_1} \qquad (1)$$

If the point (x_1, y_1) moves off to an infinite distance,

$$\text{then } \frac{1}{y_1} = \frac{1}{\infty} = 0. \qquad (2)$$

$$\text{Also, since } \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1.$$

$$\therefore \frac{x_1^2}{y_1^2} = \frac{a^2}{b^2} + \frac{a^2}{y_1^2}$$

$$= \frac{a^2}{b^2} \text{ when } y_1 = \infty$$

$$\text{Hence ultimately } \frac{x_1}{y_1} = \pm\frac{a}{b} \qquad (3)$$

Substituting from (*2*) and (*3*) in (*1*), the equation of the tangent when (x_1, y_1) is infinitely distant becomes

$$\frac{x}{a^2}\left(\pm\frac{a}{b}\right) - \frac{y}{b^2} = 0.$$

$$\text{or } \pm\frac{x}{a} - \frac{y}{b} = 0.$$

$$\text{or } \frac{x}{a} \pm \frac{y}{b} = 0. \qquad (4)$$

which represents two straight lines through the origin.

$\therefore$ The asymptotes of the hyperbola are $\frac{x}{a} \pm \frac{y}{b} = 0$.

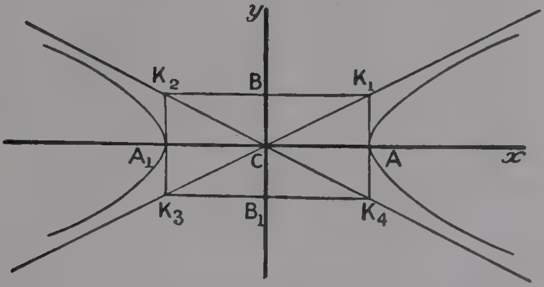
From (*4*) $y = \pm\frac{b}{a}x$.

Hence the asymptotes are two straight lines through the origin equally inclined to the axis of x.

The asymptotes are frequently written,

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=0 \quad \text{or} \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$$

164. Geometrical construction for the asymptotes.



Let **A**$_1$**A** be the transverse axis of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Along the conjugate axis measure of **CB** and **CB**$_1$ each equal to b. Through **B** and **B**$_1$ draw lines parallel to the transverse axis and through **A** and **A**$_1$ draw lines parallel to the conjugate axis. Let these meet respectively in **K**$_1$, **K**$_2$, **K**$_3$, **K**$_4$ as in the figure. Join **K**$_1$ **K**$_3$, **K**$_2$ **K**$_4$ and produce both ways. Evidently they pass through the origin and satisfy the equations:

$$y=\frac{b}{a}x \text{ and } y=-\frac{b}{a}x.$$

Hence they are asymptotes.

In the equilateral or rectangular hyperbola $a=b$ and the asymptotes become $y=x$ and $y=-x$.

These make angles of 45° with the transverse axis and are at right angles to each other.

165. The following table gives the chief properties of the Circle, Parabola, Ellipse and Hyperbola

	CIRCLE	PARABOLA	ELLIPSE	HYPERBOLA
Equation	$x^2 + y^2 = a^2$	$y^2 = 4ax$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
e		$e = 1$	$e < 1$	$e > 1$
e^2		$e^2 = 1$	$e^2 = \frac{a^2 - b^2}{a^2}$	$e^2 = \frac{a^2 + b^2}{a^2}$
CF		a	ae	ae
CZ		a	$\frac{a}{e}$	$\frac{a}{e}$
Focal distances		$\mathbf{PF} = a + x$	$\mathbf{PF} + \mathbf{PF}_1 = 2a$	$\mathbf{PF}_1 - \mathbf{PF} = 2a.$
Length Latus Rectum		$4a$	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Tangent at (x_1, y_1)	$xx_1 + yy_1 = a^2$	$yy_1 = 2a\,(x + x_1)$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$
Tangent in terms of slope	$y = mx \pm a\sqrt{1 + m^2}$	$y = mx + \frac{a}{m}$	$y = mx \pm \sqrt{m^2a^2 + b^2}$	$y = mx \pm \sqrt{m^2a^2 - b^2}$
Normal	$\frac{x}{x_1} = \frac{y}{y_1}$	$x - x_1 + \frac{2a}{y_1}(y - y_1) = 0$	$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$	$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$
Locus of foot of ⊥ from focus on tangent		$x = 0$	$x^2 + y^2 = a^2$	$x^2 + y^2 = a^2$
Locus of intersection of tangents at right angles			$x^2 + y^2 = a^2 + b^2$	$x^2 + y^2 = a^2 - b^2$
Chords of contact	$xx_1 + yy_1 = a^2$	$yy_1 = 2a\,(x + x_1)$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$
Polar of (x_1, y_1)	$xx_1 + yy_1 = a^2$	$yy_1 = 2a\,(x + x_1)$	$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$	$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$
Pole of $\mathbf{A}x + \mathbf{B}y + \mathbf{C} = 0$	$\left(-\frac{a^2}{c}.\mathbf{A}, -\frac{a^2}{c}\mathbf{B}\right)$	$\left(\frac{\mathbf{C}}{\mathbf{A}}, -\frac{2a\mathbf{B}}{\mathbf{A}}\right)$	$-\frac{\mathbf{A}}{\mathbf{C}}.a^2, -\frac{\mathbf{B}}{\mathbf{C}}.b^2$	$-\frac{\mathbf{A}}{\mathbf{C}}.a^2\ \frac{\mathbf{B}}{\mathbf{C}}.b^2$

166.—Exercises

1. Find the equations of the asymptotes to the hyperbola $9x^2 - y^2 = 36$.

2. Find the equation of the asymptotes to the hyperbola $4x^2 - 9y^2 = 1$.

3. Find the equations of the asymptotes to the hyperbola $9x^2 - 16y^2 = 144$ and the tangent of the angle between them.

4. In the hyperbola $4x^2 - 3y^2 = 12$,

Find (*a*) the transverse axis.
  (*b*) the conjugate axis.
  (*c*) the co-ordinates of the centre.
  (*d*) the eccentricity.
  (*e*) the co-ordinates of the foci.
  (*f*) the co-ordinates of the vertices.
  (*g*) the equations of the directrices.
  (*h*) the equation of the asymptotes.

5. Find the angle between the asymptotes to the hyperbola $4x^2 - y^2 = 16$.

6. Find the equation of the hyperbola whose centre is the origin and whose foci are on the x axis if $e = \frac{3}{2}$, $a = 5$.

7. Prove that the distance from an asymptote to a focus is equal to the semi-conjugate axis.

8. Find the equation of an hyperbola the vertices of which are half-way between the centre and the foci, the centre being the origin and the transverse axis lying on OX.

9. Find the equation of the hyperbola which has the lines $y = \pm \frac{3}{5}x$ for its asymptotes and the points $\pm 2, 0$ for its foci.

10. Find the equation of the hyperbola which has the asymptotes $y = \pm \frac{3}{4}x$ and passes through the point (2, 1)

11. Show that the focal distance of any point on the hyperbola is equal to the length of the straight line drawn through the point parallel to an asymptote to meet the corresponding directrix.

12. Prove that the distance of any point on the rectangular hyperbola $x^2 - y^2 = a^2$ from the centre is at mean proportional to its distances from the foci.

13. **P** is any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. **PN** is drawn perpendicular to the transverse axis and the tangent at **P** cuts the transverse axis at **T**. Prove **CA** is a mean proportional between **CN** and **CT**.

14. Prove that the product of the distances of any point on an hyperbola from its asymptotes is constant.

15. Find the length of that part of an asymptote intercepted between the directrices.

16. If a straight line drawn through the focus **S** of a hyperbola parallel to an asymptote meet the curve in **P**, prove that **SP** is one quarter of the latus rectum.

17. **P** is any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Through **P** a straight line is drawn parallel to the transverse axis and cutting the asymptotes. Prove the product of the distances of **P** from these points of intersection is equal to a^2.

18. **A**, **A_1** are the vertices of a rectangular hyperbola, **P** is any point on the curve; show that the internal and external bisectors of the angle **APA_1** are parallel to the asymptotes.

19. Prove that an asymptote and the perpendicular from the focus upon it meet on the corresponding directrix.

20. Find the equation of the following hyperbolas whose centres are at (c, o) and whose transverse axis are along the y-axis.

- (*a*) Latus rectum $=3$ and one vertex is at $(o, 2)$.
- (*b*) Conjugate axis $= 8$ and $(4, 6)$ is a point on the curve.
- (*c*) $e = 2$ and equation of the directrix is $y = 3$.
- (*d*) One focus at $(o, 6)$ and the equation of the corresponding directrix is $y = \frac{25}{6}$.

21. The distance of any point from the centre of a rectangular hyperbola varies inversely as the perpendicular distance of its polar from the centre.

22. **PN** is the ordinate of a point **P** on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. **PG** is the normal meeting the axis in **G**; if **NP** be produced to meet the asymptote in **Q**, prove that **QG** is at right angles to the asymptote.

167.—Miscellaneous Exercises

1. Deduce the equation of the conics determined by the following data; the axes of curve being the axes of the co-ordinates.

(a) $e = 1$, distance from focus to directrix $= 1$.
(b) $e = \frac{2}{3}$, distance from focus to directrix $= \frac{5}{6}$
(c) $e = \frac{3}{2}$, distance from focus to directrix $= 5$.

2. Deduce the equation of the parabola whose vertex is at the origin and focus is $(o, -4)$.

3. Deduce the equation of the ellipse with centre at origin, one focus at $(o, 3)$ and $e = \frac{3}{4}$.

4. Deduce the equation of the hyperbola with centre at origin, $a = 2$, $e = \frac{4}{3}$ and whose transverse axis is the y-axis.

5. Deduce the equation of the hyperbola whose eccentricity is 2, directrix $3x + 4y = 12$ and focus $(3, 2)$.

6. The base of a triangle is fixed in length and position.

Find the locus of the opposite vertex if

(a) the sum of the base angles is constant.
(b) the sum of the other sides is constant.
(c) the difference of the other sides is constant.

7. A tangent to the parabola $y^2 = 8x$ makes an angle of 45° with the straight line $y = 3x + 5$. Find its equation and its point of contact.

8. Prove that the straight line $y = x + \sqrt{\frac{7}{12}}$ touches the ellipse $3x^2 + 4y^2 = 1$.

9. Deduce the equation of parabola with vertex at $(-2, 4)$, $a=4$ the axis being parallel to x-axis, the curve lying on the positive side of the vertex.

10. The ordinate of the point of intersection of two tangents to the parabola $y^2 = 4ax$ is one half the sum of the ordinates of the points of contact of the tangents.

11. Find the co-ordinates of the points of contact of the tangents to the ellipse $4x^2 + 3y^2 = 5$ which are inclined at 60° to the axis of x.

12. Find the equation of a straight line which touches the curves $x^2 + y^2 = 2a^2$ and $y^2 = 8ax$.

13. Deduce the equation of the parabola whose focus is at $(2, 4)$ and directrix is $x + y = 1$.

14. Deduce the equation of an ellipse with focus at $(-2, 2)$, eccentricity $= \frac{1}{2}$ and directrix $3x - 2y + 4 = 0$.

15. Deduce the equation of an hyperbola with focus at $(-1, -2)$ eccentricity $= \frac{3}{2}$ and directrix $4x - y = 0$.

16. Find the co-ordinates of the vertex and length of the latus rectum of the following conics:

$$(a)\ y^2 = 5x + 10.$$

$$(b)\ 3x^2 + 12x - 8y = 0.$$

17. Find the co-ordinates of the focus and length of latus rectum of the following conics:

$$(a)\ 3x^2 + 4y^2 = 12.$$

$$(b)\ x^2 + 9y^2 = 16.$$

18. Find equations of directrices and asymptotes of the following conics:

$$(a)\ \frac{x^2}{4} - \frac{y^2}{16} = 1.$$

$$(b)\ 9x^2 - 36y^2 = 1.$$

$$(c)\ y^2 - 4x^2 = 1.$$

19. Write the equations of the tangent and normal to $x^2 + y^2 = 25$ at point $(3, -4)$.

20. Write the equations of the tangent and normal to $y^2 = 8x$ at point $(2, 4)$.

21. Write the equations of the tangent and normal to $9x^2 + 4y^2 = 25$ at point $(1, 2)$.

22. Write the equations of the tangent and normal to $x^2 - y^2 = 16$ at point $(-5, 3)$.

23. An ellipse and an hyperbola have the vertices of each at the foci of the other. If the equation of the ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Find that of the hyperbola.

24. For each of the conics below find the equation of the tangents making an angle of 60° with the x-axis.

(a) $x^2 + y^2 = 36.$
(b) $y^2 = 16x.$
(c) $\frac{x^2}{4} + \frac{y^2}{8} = 1.$
(d) $4x^2 - 9y^2 = 72.$

25. If perpendiculars be let fall on any tangent to the parabola $y^2 - 4ax$ from two given points on the axis equidistant from the focus, the difference of their squares will be constant.

26. Find the equations of the polars of the point (6, 4) with respect to the following conics:

(a) $x^2 + y^2 = 32.$
(b) $y^2 = 8x.$
(c) $4x^2 + y^2 = 48.$
(d) $4x^2 - y^2 = 64.$

27. Find the pole of $3x + 4y = 4$ with respect to the following conics:

(a) $x^2 + y^2 = 36.$
(b) $y^2 = 3x.$
(c) $6x^2 + 8y^2 = 16.$
(d) $15x^2 - 3y^2 = 9.$

28. If the vertices of an hyperbola lie two-thirds of the distances from the centre to the foci, find the angles between the transverse axis and the asymptotes.

29. If **FQ** be the perpendicular from the focus of the parabola $y^2 = 4ax$ on the normal at **P**, then **FQ**^{2} = **AN·PF** if **N** is the foot of ordinate of **P**.

30. The tangent at **P** on an ellipse cuts the axis major in **T**. The normal at **P** cuts the axis major in **G**. Find the position of **P** that **PT** and **PG** may be equal.

31. Find the equation of the chord of contact of the point indicated for the following conics:

(a) $x^2 + y^2 = 25$ at point (4, 7).
(b) $y^2 = 6x$ at point (2, 7).
(c) $4x^2 + 6y^2 = 50$ at point (3, 8).
(d) $2x^2 - 5y^2 = 40$ at point (1, 4).

32. In the equilateral hyperbola if PG the normal at P, meet the transverse axis in G then PCG is an isosceles triangle.

33. Find the equations of the tangents to $x^2 + y^2 = 25$ which are parallel to $3x + 4y = 8$.

34. Find the equation of the tangent to $y^2 = 8x$ which is perpendicular to $3x + 4y = 8$.

35. Find the equations of the tangents to $4x^2 + 9y^2 = 25$ which are parallel to $x + 2y - 7 = 0$.

36. Find the equations of the tangents to $4x^2 - 9y^2 = 25$ which are perpendicular to $2x - 3y = 5$.

37. Find the equation of the normal to $8x^2 + 9y^2 = 72$ which makes an angle of 45° with the x-axis.

38. Find the equations of the tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are parallel to the line through the positive extremities of the axes.

39. Find the equation of the chord of the hyperbola $x^2 - 2y^2 = 4$ through (3, 1) which is bisected by the diameter $2y = x$.

40. Find the eccentricity of an ellipse if the normal at one extremity of the latus rectum passes through the opposite end of the minor axis.

41. Find the equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ which passes through the point (2, 1) and is bisected at that point.

42 Prove that an hyperbola and an ellipse having the same foci cut each other at right angles.

43. If QVQ_1 be a double ordinate of the diameter CP of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b} = 1$, and if the tangent at Q meet CP in T then $CV \cdot CT = CP^2$.

44. A straight line has its extremities on two fixed straight lines and cuts off from them a triangle of constant area; find the locus of the middle point of the line.

45. If normals be drawn at the extremities of any focal chord of an ellipse, a line through their intersection parallel to the major axis will bisect the focal chord.

46. **A**, **A**$_1$ are the vertices of a rectangular hyperbola, and **P** is any point on the curve; show that the internal and external bisectors of the angle **APA**$_1$ are parallel to the asymptotes.

47. If a straight line cut an hyperbola in **Q**, **Q**$_1$ and its asymptotes in **R**, **R**$_1$ show that the middle point of **QQ**$_1$ will be the middle point of **RR**$_1$.

48. The circle $x^2 + y^2 = 64$ intersects the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$ in four points; prove that the product of the distances of the four points of intersection from one asymptote is equal to the product of their distances from the other.

49. By means of ruler, compass, etc. construct:
(*a*) Parabola
(*b*) Ellipse
(*c*) Hyperbola

50. By means of ruler, compass, etc. plot on square paper the following conics and draw a tangent to each at the point indicated:
(*a*) $x^2 + y^2 = 25$ at point (3, 4).
(*b*) $y^2 = 8x$ at point (2, 4).
(*c*) $4x^2 + y^2 = 17$ at point (2, 1).
(*d*) $4x^2 - 3y^2 = 13$ at point (2, 1).

ANSWERS

§ 6. (Page 5.)

4. $(0, 0), (2a, 0), (a, a\sqrt{3})$.

5. $(0, 0), (b, 0), (b, b), (0, b)$.

§ 16. (Page 11.)

4. 13·04.

5. 10, 17, 9.

6. (*a*) $\sqrt{58}, \sqrt{82}, 10$;
(*b*) $\frac{1}{2}\sqrt{234}, 3\sqrt{5}, \frac{1}{2}\sqrt{306}$.

7. $\sqrt{82}, 2\sqrt{10}, \sqrt{85}, \sqrt{29}$;
$7\sqrt{2}, \sqrt{137}$.

8. (0, 0) – the origin.

9. $(\frac{8}{3}, \frac{7}{3})$ and $(\frac{13}{3}, \frac{5}{3})$.

11. $\sqrt{a^2+b^2}$.

12. $\frac{4}{9}\sqrt{65}$.

13. 3 or – 13.

14. $4x-10y+29=0$.

15. $(\frac{103}{62}, \frac{221}{62})$; 3·7 nearly.

16. (7, 2), $(-\frac{245}{37}, \frac{158}{37})$.

20. $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

21. $(\frac{31}{5}, \frac{61}{5})$.

22. (–18, –14).

§ 19. (Page 16.)

3. 36·5.

4. 25·5.

8. 36 miles.

9. 1 : 3.

§ 22. (Page 21.)

2. $y=5x$.

3. (*a*) The axis of x; (*b*) The axis of y.

4. $x=4$.

5. $3x-5y=17$.

6. $2x+y+5=0$.

7. $x^2+y^2-8x-6y=0$.

8. $x^2+y^2-2x+4y=31$.

10. $x-2y+3=0$.

§ 24. (Page 23.)

1. (*a*) (2, 7); (*b*) (3, 2); (*c*) (2, –3); (*d*) (8, –6) and (–8, 6); (*e*) (5, 12) and $(-\frac{220}{17}, \frac{21}{17})$.

2. (0, 9) and (–15, 0).

3. (0, 0) and (6, 0).

4. $2x=a$.

5. $7x+4y=20$. ?

6. $x^2+y^2=9$.

§ 27. (Page 27.)

2. (*a*) $\frac{x}{5}+\frac{y}{2}=1$; (*b*) $\frac{x}{4}+\frac{y}{6}=-1$; (*c*) $\frac{x}{3}-\frac{y}{8}=1$.

3. (*a*) $x-3y=0$; (*b*) $9x-2y+13=0$; (*c*) $8x+11y+34=0$.

   Intercepts :—(*a*) 0, 0; (*b*) $-\frac{13}{9}$, $\frac{13}{2}$; (*c*) $-\frac{34}{8}$, $-\frac{34}{11}$.

4. $(-5,\ -3\frac{1}{3})$.

5. $(1\frac{7}{8},\ 1\frac{13}{28})$.

6. $(1\frac{1}{3},\ 4)$.

7. Ratio of equality.

8. Sides, $x+9y+14=0$; $5x+3y+28=0$, $2x-3y-14=0$;

   Medians, $x-5y=14$, $x+2y=-7$, $3x-y=0$;

   Centroid $(-1,\ -3)$.

9. Sides, $2x-5y=-24$, $3x+2y=2$, $x-y=4$, $3x+4y=33$;

   Diagonals, $8x-y=18$, $x+9y=34$.

10. $(-6,\ -7)$, $(5,\ -2)$, $(-3,\ 4)$.

§ 40. (Page 38.)

3. (*a*) $x=\sqrt{3}y$; (*b*) $y+x\sqrt{3}=0$; (*c*) $5x-7y+35=0$; (*d*) $3y\pm 4x+9=0$; (*e*) $x-y=1$.

4. (*a*) $9,\ -\frac{1}{2},\ -\frac{12}{5}$); (*b*) $(\frac{5}{3},\ 0)$.

5. $(3,\ 3\frac{1}{2})$.

6. $x+y+1=0$.

7. $\frac{5}{29}\sqrt{58}$; $\tan^{-1}\frac{7}{3}$.

8. (*a*) $\mathbf{C}=0$; (*b*) $\mathbf{A}=0$; (*c*) $\mathbf{B}=0$; (*d*) $\mathbf{A}=\mathbf{B}$; (*e*) $\mathbf{A}+\mathbf{B}=0$.

9. $m=1$.

10. $m=-\frac{1}{9}$, $b=2\frac{7}{9}$.

11. $a=8$, $b=-4$.

13. Intercept $=\frac{\mathbf{C}'-\mathbf{C}}{\mathbf{B}}$.

§ 44. (Page 42.)

1. (*a*) 45°; (*b*) 30°; (*c*) 90°; (*d*) $\tan^{-1}5$.

3. $9x+4y+47=0$.

5. $2x+3y=14$.

6. $7x+5y=4$.

7. $5x-3y+8=0$.

8. $x-y+2=0$ and $x+y-12=0$.

9. $x-\sqrt{3}y=3\sqrt{3}-5$ and $x+\sqrt{3}y=-3\sqrt{3}-5$.

10. $(2\frac{1}{7},\ -4\frac{1}{7})$.

11. $\left(b,\ \frac{ab-b^2}{c}\right)$.

12. 12 : 5.

14. $(17\sqrt{3}-16)x+47y=344+34\sqrt{3}$; and $(17\sqrt{3}+16)x-47y=34\sqrt{3}-344$.

15. $6x+y+8=0$: and $x-6y=11$.

17. $\left(\frac{a}{2},\ \frac{b^2+c^2-ab}{2c}\right)$.

18. $\mathbf{B}x-\mathbf{A}y+p\sqrt{\mathbf{A}^2+\mathbf{B}^2}=0$.

§ 57. (Page 55.)

(*a*)

1. (*a*) $\frac{38}{\sqrt{29}}$; (*b*) $\frac{24}{\sqrt{13}}$; (*c*) $\frac{12\sqrt{2}}{5}$; (*d*) $\frac{5}{\sqrt{41}}$; (*e*) $\frac{35}{\sqrt{74}}$.
2. $1\frac{2}{5}$.
3. $\frac{c_2-c_1}{\sqrt{a^2+b^2}}$
4. $(-24, 55)$.
5. $8x+6y=15$.
6. $x-y=0$.
7. $2\frac{7}{13}$.
8. $(m_1-m)(ax-by)+b(m_1c-mc_1)+a(c_1-c)=0$.
10. $abc-af^2-bg^2-ch^2+2fgh=0$.
11. $(b-m_1a-c_1)\ (y-mx-c)=(b-ma-c)\ (y-m_1x-c_1)$.
12. $(\mathrm{AC}_1-\mathrm{A}_1\mathrm{C})x+(\mathrm{BC}_1-\mathrm{B}_1\mathrm{C})y=0$.
13. $\sqrt{2}$.
14. $\frac{\pm 2\sqrt{130}-11}{7}$
15. $x+21y=6$ and $189x-9y=692$.
17. $3x+y=2$ and $x-3y=24$.
19. $(\frac{341}{238}, \frac{433}{238})$.
20. $33x+61y=216$.
21. $11\frac{(7\sqrt{3}+11)}{13}$.
22. $\frac{6}{5}\sqrt{2}$.
24. $\frac{ab(c-d)}{\sqrt{a^2+b^2}}$.
25. $15x+14y=100$.
26. $13x-7y=3$ and $7x+13y=119$.
27. $(24+13\sqrt{3})x+23y+52\sqrt{3}+257=0$ and $(24-13\sqrt{3})x+23y-52\sqrt{3}+257=0$.
28. $5x-12y+56=0$ and $5x-12y-74=0$.
29. $6x+y=31$, $x-y+3=0$; $\frac{14}{\sqrt{37}}$, $\frac{7}{\sqrt{2}}$.
30. $(5\frac{10}{13}, 3\frac{9}{13})$.
32. $12x-5y=26$, $y=2$.
33. $\mathrm{A}x+\mathrm{B}y+p\sqrt{\mathrm{A}^2+\mathrm{B}^2}=0$.
34. $\mathrm{B}x-\mathrm{A}y+\mathrm{A}k-\mathrm{B}h=0$.
35. (*a*) $(\frac{306}{121}, -\frac{87}{121})$; (*b*) $(-\frac{175}{121}, -\frac{61}{121})$.

(*b*)

1. $2x^2-11xy+12y^2$.
2. $x^2+xy=0$.
3. $(-\frac{5}{2}, \frac{9}{2})$; $2(x^2+y^2)=19$.
5. $(\mathrm{A}\sqrt{3}+\mathrm{B})x+(\mathrm{B}\sqrt{3}-\mathrm{A})y+2\mathrm{C}=0$.
6. $2xy+a^2=0$.
8. $9x^2-25y^2=0$.
9. $\tan^{-1}\frac{4}{3}$; $3x^2+2y^2=10$.

§ 58. (Page 60.)

1. (*a*) $\sqrt{145}$; (*b*) $\sqrt{5a^2+16ab+13b^2}$; (*c*) $a+b$.
2. $\mathrm{P}(\frac{5}{8}, 3\frac{3}{8})$; $\mathrm{Q}(17\frac{1}{2}, 16\frac{1}{2})$.
3. $\frac{3}{2}ab$.
4. $41\frac{1}{2}$.
5. $10\frac{3}{5}$.
6. $x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0$.
7. $8x+7y=5$.

9. $2ax + 2by = c + d$.
10. $x = y \tan a + b$.
13. $x \cos a + y \sin a = a$.
14. $\tan^{-1} \frac{126}{41}$.
15. $m = \frac{3}{4}$, $b = -4\frac{1}{4}$.
16. $x + y\sqrt{3} + 2(\sqrt{3} - 1) = 0$.
17. $\dfrac{mh - k + b}{\sin a - m \cos a}$.
18. $\dfrac{x}{a} + \dfrac{y}{b} = \dfrac{h}{a} + \dfrac{k}{b}$.
20. $bx + ay = ad$.
21. $(-2, -8)$ and $(-6, -14)$.
23. $a = 6$ or -4; $(\frac{1}{2}, \frac{1}{3})$, $(\frac{1}{2}, -\frac{1}{2})$.
25. 45°.
26. $65x - 65y + 14a + 3b = 0$.
27. $b(am + c)x + (bd - ab + adm + ac)y - bd(am + c) = 0$.
29. $\dfrac{ab}{\sqrt{a^2 + b^2}}$.
30. $x + 4y + 1 = 0$.
31. $x + 7y + 6 = 0$.
32. $(6\frac{73}{187}, 5\frac{124}{187})$; $(-3\frac{9}{77}, \frac{69}{77})$; $(26\frac{26}{119}, 4\frac{67}{119})$; $(0, 57)$.
33. $(4\frac{7}{10}, -\frac{1}{10})$.
35. $x + 6y = 39$ and $6x - y = 12$.
38. $2x + 3y + 11 = 0$.
39. $165x + 605y + 3114 = 0$; $332x - 747y + 3466 = 0$; $497x - 142y - 2178 = 0$.
40. $(\frac{2}{3}, \frac{2}{3})$.
41. (a) $(ah - cf)x + (bh - cg)y = 0$; (b) $(ag - bf)(x - y) + c(f + g) - h(a + b) = 0$.
42. $5x - 14y = 6$ and $154x + 55y = 229$.
43. $49x - 245y = 242$.
45. $x + y = 10$.
46. $x - y = 10$, or $y - x = 10$.
47. $7x + 5y + 50 = 0$, and $9x + 35y + 150 = 0$.
48. $x - 4y = 11$.
49. $2(a - h)x + 2(b - k)y = a^2 + b^2 - h^2 - k^2$.
50. $2x + y + 15 = 0$.
51. 45°.
52. $10x + 4y + 11 = 0$, and $4x - 10y - 33 = 0$.
53. $(11, 1)$, $(-1, -5)$, or $(-5, 7)$.
55. $-\frac{66}{31}$.
56. 24.

§ 64. (Page 69.)

1. $x^2 + y^2 = 5$.
2. $(x - 6)^2 + (y - 2)^2 = 9$.
3. $(x + 5)^2 + (y + 1)^2 = 26$.
4. $a^2 + b^2 = c^2$.
5. (a) $(3, 1)$, 5; (b) $(-\frac{7}{8}, -\frac{5}{8})$, $\dfrac{\sqrt{330}}{8}$; (c) $(7, 0)$, 7; (d) $(0, -b)$, $\sqrt{b^2 + c^2}$.
8. $x^2 + y^2 - x - 7y = 0$.
9. $x^2 + y^2 - 3x = 19$.
11. (a) $f = 0$; (b) $g = 0$.
12. $x^2 + y^2 - 4x + 4y = 2$.
13. $x^2 + y^2 - 4x + 6y = 16$.
15. $47(x^2 + y^2) - 181x + 341y - 1996 = 0$.
16. $\sqrt{10}$.
17. $14\sqrt{2}$.
21. $2x - 5y = 15$.
22. 123.
23. $14(x + y) = 17$.
24. $c = c'$.
25. $(h + k)(x^2 + y^2) - (h^2 + k^2)(x + y) = 0$.

§ 70. (Page 77.)

1. $3x + 5y = 34.$
2. $y = x \pm \sqrt{70}.$
3. (a) $\mathsf{A}x + \mathsf{B}y = \pm r\sqrt{\mathsf{A}^2+\mathsf{B}^2};$ (b) $\mathsf{B}x - \mathsf{A}y = \pm r\sqrt{\mathsf{A}^2 + \mathsf{B}^2}.$
4. $(2, 4).$
5. $a^2b^2 = r^2(a^2+b^2).$
6. $c = g^2.$
7. $x^2+y^2-2(7\pm2\sqrt{5})(x+y)+69\pm28\sqrt{5}=0.$
8. $4\sqrt{\frac{313}{29}}.$
11. $x-3y=3;$ $\left(\dfrac{21\pm3\sqrt{119}}{10}, \dfrac{-3\pm\sqrt{119}}{10}\right).$
12. $(\mathsf{A}^2+\mathsf{B}^2)(x^2+y^2)=\mathsf{C}^2.$

Use p. 4

13. $34(x^2+y^2)-476x-136y+1753=0.$
14. $x-\sqrt{3}y\pm10=0.$
15. $3x-4y+11=0;$ $12x+5y=96;$ $gx+fy=0.$
17. $c^2=r^2(\mathsf{A}^2+\mathsf{B}^2).$

§ 81. (Page 85.)

1. (a) $3x+5y=30;$ (b) $ax=r^2;$ (c) $4x+17=0;$ (d) $10x-2y=7;$ (e) $hx+ky=h^2+k^2-r^2.$
2. (a) $(2, -7);$ (b) $(-\frac{23}{12}, \frac{23}{6});$ (c) $(-35, 11);$ (d) $(0, 0).$
3. (c) $3x-4y+25=0;$ (e) $9x+13y=25;$ (f) $13x-9y=0;$ (g) $24x-7y=125.$
4. $\left(\dfrac{c^2}{a}, \dfrac{c^2}{b}\right).$

§ 83. (Page 87.)

1. (a) 6; (b) 5; (c) 8; (d) $\sqrt{c}.$
2. $x^2+y^2-10x-4y=7.$
3. $x^2+y^2-16x+51=0.$
4. $4x+3y=25$ and $3x-4y=25$

MISCELLANEOUS EXERCISES. (Page 87.)

1. $3x+4y=25.$
2. $2x-11y+329=0.$
3. 113.
4. 29.
5. $\dfrac{x}{h}+\dfrac{y}{k}=2.$
6. (a) $\dfrac{x-h}{x_1-x_2}=\dfrac{y-k}{y_1-y_2};$ (b) $(x_1-x_2)(x-h)+(y_1-y_2)(y-k)=0.$
8. 7:4.
9. $(\frac{283}{115}, -\frac{141}{115}).$
10. 77.
12. $x^2+y^2=12.$
13. $(a+c, b+d).$
15. $x^2+y^2-6x-3y=0.$
16. $3x-y=3k\pm k\sqrt{10}.$
17. $(\frac{382}{147}, \frac{302}{147}).$
18. $x^2+y^2=hx.$
19. $x^2+y^2-ax-by=0.$
20. $x^2+y^2-gx-fy=0.$
21. $(\frac{12}{13}, \frac{21}{13}).$
22. $8(x^2+y^2)-25x+75y+71=0.$

23. 10 or $\frac{6}{7}$.
24. (a) $(7, 1)$; (b) **AD = DB.**
26. $\frac{238}{143}$.
27. $x^2+y^2-2h(x-a)-2k(y-b) = a^2+b^2$.
28. $2x-(b+c)y+2abc=0$.

EXERCISE 92. (Page 97).

2. $(-a, o)$, $4a$; $(o, -a)$, $4a$; $(\frac{1}{64}, 0)$, $\frac{1}{16}$.
3. $y^2 = 4a(x-a)$.
4. $x^2 = 4ay$.
5. $y^2 = 4a(x+a)$.
6. $(5a, 2a\sqrt{5})$. $(5a, -2a\sqrt{5})$.
7. $(a, 2a)$ $\left(\frac{a}{9}, -\frac{2a}{3}\right)$.
8. $(\frac{3}{4}, 3)$ $(\frac{4}{3}, 4)$.
9. $a = -2$.
10. $8a\sqrt{3}$.
11. $6\sqrt{5}$.
12. $x^2+y^2 = 5ax$.
13. $(x-2)^2+(y-3)^2 = \frac{(x+y-2)^2}{2}$; $(x-3)^2 + (y-5)^2 = \frac{(24x+7y+18)^2}{625}$
$(x-3)^2+(y+3)^2 = \frac{(5x+12y-13)^2}{169}$
$(x-c)^2+(y-d)^2 = \frac{(kx+ly+m)^2}{k^2+l^2}$.
14. $(0, 0)$; $3x+4y+5=0$.
$(2, 3)$; $5x+12y+7=0$.

EXERCISE 95. (Page 102).

1. $(0, 0)$; $(2, 0)$; $y = 0$; $x+2=0$.
$(2, 1)$; $(3, 1)$; $y = 1$; $x = 1$.
$(-\frac{11}{8}, 3)$; $(\frac{5}{8}, 3)$; $y-3=0$; $8x+27=0$.
$(0, -2)$; $(0, -3)$; $x = 0$, $y+1=0$.
$(-a, -c)$; $(-a, -c-b)$; $x+a=0$; $y+c=b$.
$(\frac{1}{2}, -2)$; $(2, -2)$; $y+2=0$; $x+1=0$.
$(-3, 11)$; $(-3, \frac{43}{4})$; $x+3=0$; $y-\frac{45}{4}=0$.
$(1, -\frac{1}{2})$; $(\frac{13}{16}, -\frac{1}{2})$; $y = -\frac{1}{2}$; $x = \frac{19}{16}$.
$(\frac{3}{2}, \frac{9}{2})$; $(\frac{3}{2}, \frac{35}{8})$; $x-\frac{3}{2}=0$; $y = \frac{37}{8}$.
2. $2y^2-12y+9x=0$.
3. $12x^2+36x+y+25=0$.
4. $y^2-4y-8x+28=0$.
5. $y^2 = 4ax+4a^2$.
6. $y^2 = 4ax-4x^2$.
7. $y^2-6y+4x+9=0$.
8. $y^2+10y+20x+65=0$.
9. $y^2+4y-8x+12=0$.
10. $x^2+4y-8=0$.
11. $x^2-2xy+y^2-4ay-2a^2=0$.
12. $x^2+y^2-2x-4y=0$.
13. $x^2 = 2y$.

EXERCISE 98. (Page 105).

2. 2′.
3. 20′.
4. 28′.
5. $27\frac{7}{8}'$.
6. 4′, 4.32′, 5.28′, 6.88′, 9.12′.
9. $4a(2 \pm \sqrt{3})$.
10. $x^2-2x+y^2-6y=0$.

EXERCISE 105. (Page 112).

1. $x-y+1=0$; $x+y-3=0$.
$x-y-2=0$; $x+y-6=0$.
$3x-2y+3=0$; $2x+3y-11=0$.
$2x-5y-5=0$; $5x+2y-27=0$.
$2x+4y-1=0$; $8x-4y-9=0$.

2. $y=\frac{x}{\sqrt{3}}+\frac{3\sqrt{3}}{4}$; $y=x+\frac{3}{4}$; $y+x+\frac{3}{4}=0$.
$x\sqrt{3}=y+6$; $x\sqrt{3}+y+6=0$; $x\sqrt{3}+3y+2=0$.

3. $y=x+a$; $y+x+a=0$.

4. $y=x+3$; $y+x+3=0$.

5. $y(2+2\sqrt{2})=2x+9+6\sqrt{2}$.
$y(2-2\sqrt{2})=2x+9-6\sqrt{2}$.

6. $K=1$.

7. $p=-\frac{1}{4}$.

8. $y=\frac{(5\pm\sqrt{21})x}{2}+\frac{2}{5\pm\sqrt{21}}$.

9. $y=\frac{1\pm\sqrt{2}}{2}x+\frac{1}{1\pm\sqrt{2}}$

10. $9x+6y+8=0$.

11. $9x+12y+32=0$.

12. $x+y-15=0$; $x-y-15=0$.

13. Points of Contact: $(14-4\sqrt{10},\ -4+2\sqrt{10})$; $(14+4\sqrt{10},\ -4\ -2\sqrt{10})$.

15. $a^{\frac{1}{3}}x+b^{\frac{1}{3}}y+a^{\frac{2}{3}}b^{\frac{2}{3}}=0$.

16. $3^{\frac{1}{3}}x+2^{\frac{1}{3}}y+3^{\frac{2}{3}}\,2^{\frac{2}{3}}=0$.

EXERCISE 110. (Page 117).

1. 32; 4; 6.

2. 2; $2\sqrt{2}$; $2\sqrt{3}$.

5. $\frac{a}{2}(1+\sqrt{5})$, $a\sqrt{2+2\sqrt{5}}$.

11. $(y-k)\,k=2a\,(x-h)$

12. $y\,(y-y_1)=2a\,(x-x_1)$.

13. $y^2=2a\,(x-a)$.

EXERCISE 117. Page 123).

1. $2x-5y+6=0$; $4x-3y+8=0$; $3x+y+9=0$.

2. $\left(\frac{17}{2},\ -\frac{3}{2}\right)$; $\left(-2,\ -\frac{4}{3}\right)$; $\left(-\frac{7}{6},\ -1\right)$.

3. $\left(-a,\ -\frac{2pa}{b}\right)$.

5. $\left(-2a-at^2,\ -\frac{2a}{t}\right)$; $y^2\,(x+2a)+4a^3=0$.

7. $2x-3y+4=0$; $\sqrt{13}$.

EXERCISE 126. (Page 135).

1. $(x-1)^2+y^2=\dfrac{(x+y)^2}{8}$.

2. $\dfrac{\sqrt{5}}{3}$; $\dfrac{8}{3}$; $(\sqrt{5}, 0)$; $(-\sqrt{5}, 0)$;
$\dfrac{\sqrt{5}}{3}$; $\dfrac{8}{3}$; $(0, \sqrt{5})$; $(0 - \sqrt{5})$.

3. $\dfrac{\sqrt{2}}{\sqrt{3}}$; $\dfrac{1}{\sqrt{2}}$; $\dfrac{1}{2}$; $\left(\dfrac{1}{\sqrt{6}}, 0\right)$;
$\left(\dfrac{-1}{\sqrt{6}}, 0\right)$.

4. $\frac{1}{2}$, $-\frac{1}{3}$; $\frac{1}{2}$, $\frac{11}{3}$; $\frac{1}{2}$, $-\frac{13}{3}$;
$\dfrac{\sqrt{7}}{4}$; $\dfrac{1}{2}$, $\sqrt{7}-\dfrac{1}{3}$;
$\dfrac{1}{2}$, $-\sqrt{7}-\dfrac{1}{3}$.

5. $\dfrac{1}{\sqrt{2}}$; $\left(\dfrac{3}{4}, \pm\dfrac{3}{4}\right)$; $y=\pm\dfrac{3}{2}$.

6. $\dfrac{(x-a)^2}{a^2}+\dfrac{y^2}{b^2}=1$.

7. $\dfrac{x^2}{a^2}+\dfrac{(y-b)^2}{b^2}=1$.

8. $\dfrac{\sqrt{55}}{\sqrt{7}}$; $\dfrac{\sqrt{55}}{\sqrt{3}}$.

10. $x^2+4y^2-4x+24y+24=0$.

11. $20x^2+24xy+27y^2+72(x-y+1)=0$.

12. $36x^2+25y^2-72x+50y+60=0$.

EXERCISE 127. (Page 136).

1. (a) $\dfrac{1}{\sqrt{3}}$; $\left(\dfrac{1}{\sqrt{6}}+1, -2\right)$;
$\left(-\dfrac{1}{\sqrt{6}}+1, -2\right)$; $\dfrac{2\sqrt{2}}{3}$
(b) $\dfrac{1}{2}$; $\left(1, \dfrac{1}{2\sqrt{6}}-1\right)$;
$\left(1, -\dfrac{1}{2\sqrt{6}}-1\right)$; $\frac{1}{4}\sqrt{6}$.
(c) $\dfrac{1}{\sqrt{5}}$; $\left(1+\dfrac{\sqrt{6}}{\sqrt{5}}, 2\right)$;
$\left(1-\dfrac{\sqrt{6}}{\sqrt{5}}, 2\right)$; $\dfrac{48}{5\sqrt{6}}$.

2. $\dfrac{x^2}{36}+\dfrac{y^2}{20}=1$

3. $\dfrac{x^2}{16}+\dfrac{y^2}{12}=1$.

4. $9x^2+8y^2+16y-64=0$

5. $\dfrac{x^2}{16}+\dfrac{y^2}{25}=1$.

6. $9x^2+25y^2=225$.

7. $\dfrac{9(x+2)^2}{128}+\dfrac{(y-3)^2}{16}=1$.

8. $\dfrac{4x^2}{81}+\dfrac{y^2}{18}=1$.

10. $\frac{1}{2}$; $3x^2+4y^2=3a^2$.

11. $e=\dfrac{1}{\sqrt{2}}$.

12. $4x^2+9y^2=4a^2$; $\frac{1}{3}\sqrt{5}$.

13. $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$.

Exercise 136. (Page 146).

1. $c = \pm\sqrt{\frac{5}{6}}$.
2. $\frac{x}{4} + \frac{y}{2} = 1$; $4x - 2y = 1$.
3. $y = \sqrt{3}x \pm \sqrt{3a^2 + b^2}$.
4. $y = x \pm \sqrt{a^2 + b^2}$; $y = -x \pm \sqrt{a^2 + b^2}$.
5. $y = x \pm \sqrt{7}$; $y = -x \pm \sqrt{7}$.
7. 20, 10.
8. $\pm\sqrt{20}$, ± 13.
9. The axis minor produced.
15. $y = -\frac{b}{a}x \pm b\sqrt{2}$.
17. $x \pm y = \pm\sqrt{a^2 + b^2}$.
24. The foci are 1·3 inches from the centre; ·706, ·662.

Exercise 139. (Page 150).

1. $\frac{3x}{a^2} + \frac{5y}{b^2} = 1$; $9x + 8y = 0$; $4x + 21y = 12$.
2. 8, 27; $\frac{3}{2}$, 6; $9a$, 4.
3. $\frac{4}{a}$, $\frac{25}{b}$.
4. $\frac{-41}{n}$, $\frac{-16m}{n}$.
5. $\frac{a^4}{x_1(a^2 - b^2)}$, $\frac{-b^4}{y_1(a^2 - b^2)}$.
6. $4x + 63y = 25$.
7. $3y - x\sqrt{3} = \sqrt{13}$.

Exercise 144. (Page 157).

3. (a) $\frac{\sqrt{7}}{2}$; $(\sqrt{7}, 0)$; $(-\sqrt{7}, 0)$; 3.
   (b) $\sqrt{6}$; $\left(\frac{\sqrt{54}}{\sqrt{5}}, 0\right)$; $\left(\frac{-\sqrt{54}}{\sqrt{5}}, 0\right)$; $6\sqrt{5}$.
4. (a) $a = 3$, $b = 2$; $\frac{8}{3}$; $(\pm 3, 0)$; $(\pm\sqrt{13}, 0)$; $x + \frac{9}{\sqrt{13}} = 0$; $x - \frac{9}{\sqrt{13}} = 0$.
   (b) $a = 4$, $b = 2\sqrt{3}$; 6; $(\pm 4, 0)$; $(\pm 2\sqrt{7}, 0)$; $x + \frac{8}{\sqrt{7}} = 0$; $x - \frac{8}{\sqrt{7}} = 0$.
5. (a) $\frac{x^2}{4} - \frac{3y^2}{4} = 1$.
   (b) $\frac{x^2}{9} - \frac{y^2}{3} = 1$.
   (c) $\frac{x^2}{16} - \frac{y^2}{20} = 1$.
   (d) $\frac{x^2}{36} - \frac{y^2}{9} = 1$.
   (e) $\frac{x^2}{1} - \frac{y^2}{3} = 1$.
6. 16; $\frac{1}{4}$.
7. (a) $\frac{x^2}{16} - \frac{y^2}{9} = 1$.
   (b) $\frac{x^2}{36} - \frac{y^2}{108} = 1$.
   (c) $7x^2 - 9y^2 = 81$.
8. (a) $\frac{(x+1)^2}{16} - \frac{(y-3)^2}{25} = 1$;

$(-1+\sqrt{41}, 3)$;

$(-1-\sqrt{41}, 3)$; $\frac{25}{2}$.

9. $(x+1)^2+(y+2)^2 =$
$4(y-4)^2$.

10. $\frac{x^2}{144}-\frac{y^2}{180}=1$.

11. $(x-3)^2+(y-4)^2 =$
$\frac{9}{4}\frac{(4x+6y-15)^2}{52}$

13. $(5, -\frac{20}{3})$.

EXERCISE 149. (Page 163)

2. $x^2-y^2=\frac{9}{2}$.

3. $11x^2-25y^2=275$.

4. (*a*) $\frac{x^2}{9}-\frac{y^2}{16}=1$.
   (*b*) $25x^2-144y^2=900$.
   (*c*) $65x^2-36y^2=441$.
   (*d*) $x^2-y^2=32$.

5. $\frac{x^2}{55}-\frac{y^2}{9}=1$.

6. $\frac{x^2}{a^2}-\frac{y^2}{3a^2}=1$.

8. $\sqrt{(x+10)^2+(y+10)^2}-$
$\sqrt{(x-10)^2+(y-10)^2}=20$.

9. $9x^2-16y^2=3600$.

10. $\frac{1+\sqrt{5}}{2}$.

11. $2x^2-y^2=5$.

EXERCISE 153. (Page 169).

1. (*a*) $\sqrt{3}x-y=\sqrt{3}$;
   $x+\sqrt{3}y=13$.
   (*b*) $3x-\sqrt{6}y=6$;
   $2x+\sqrt{6}y=9$.
   (*c*) $-8x-3\sqrt{10}y=1$;
   $-15x+4\sqrt{10}y=50$.

2. $y\sqrt{3}=x\pm\sqrt{6}$; $y=x\pm 2\sqrt{2}$.

3. $\pm x\sqrt{228}-16y=12$.

4. (*a*) $8x+y\sqrt{10}=6$;
   $3x\sqrt{10}-24y=30\sqrt{10}$.
   (*b*) $x\sqrt{66}-y\sqrt{30}=10$;
   $\frac{30x}{\sqrt{66}}+y\sqrt{30}=22$.

5. $m=\pm\frac{3}{2}$.

6. $m=\pm\frac{2}{3}$.

7. $5A^2=2B^2$.

8. $x+y\sqrt{2}=10$.

9. $x3\sqrt{3}+y\sqrt{39}=\frac{13\sqrt{13}}{3}$;
$x3\sqrt{3}-y\sqrt{39}=\frac{13\sqrt{13}}{3}$.

10. $(2, 1)$; $(-16, 13)$.

11. $3x+4y=10$;
$234x-192y+6740=0$.

12. $2x+3y\pm 3\sqrt{7}=0$.

13. *tan* $-1\,\frac{40}{9}$.

EXERCISE 158. (Page 174).

1. $x^2 - y^2 = 8.$

2. (b) $4x - 3\sqrt{3}\cdot y = 6.$

   (c) $3\sqrt{3}\cdot x + 4y = 26\sqrt{3}.$

3. $\theta = \tan^{-1} \dfrac{2\sqrt{13}}{\sqrt{15}}.$

7. $\dfrac{m\sqrt{13} + \sqrt{4m^2 - 9}}{\sqrt{m^2 + 1}}$; $\dfrac{-m\sqrt{13} + \sqrt{4m^2 - 9}}{\sqrt{m^2 + 1}}$; 9.

8. $x^2 + y^2 = \frac{10}{3}.$

9. $x^2 + y^2 = 9.$

10. $x^2 + y^2 = \frac{5}{3}.$

11. $y = \pm\sqrt{5};\ x = \pm\sqrt{5}.$

12. $x - y = 2.$

13. $125x - 48y = 481.$

14. $y = \pm x \pm \sqrt{a^2 - b^2}.$

EXERCISE 161. (Page 178).

1. $\dfrac{3x}{a^2} - \dfrac{5y}{b^2} = 1;\ \dfrac{x}{4} - \dfrac{2y}{3} = 1;$

   $\dfrac{2x}{9} - \dfrac{5y}{4} = 1.$

2. $\dfrac{-4}{5}, \dfrac{9}{5};\ 3, \dfrac{-2}{3};\ \dfrac{9a}{c}, -\dfrac{4}{c}.$

3. $\dfrac{4}{a}, \dfrac{-25}{b}.$

4. $\dfrac{-l}{n}\cdot a^2,\ \dfrac{m}{n}\cdot b^2.$

5. $\dfrac{a^4}{x_1(a^2 + b^2)}, \dfrac{b^4}{y_1(a^2 + b^2)}.$

6. $2x - 27y = 10.$

7. $\dfrac{9}{a}, \dfrac{7}{a}.$

EXERCISE 166. (Page 183).

1. $y = \pm 3x.$

2. $2x \pm 3y = 0.$

3. $3x + 4y = 0;\ \ 3x - 4y = 0;$

   $Tan^{-1}\frac{24}{7}$ or $Tan^{-1} -\frac{24}{7}.$

4. $2\sqrt{3};\ 4;\ 0, 0;\ \dfrac{\sqrt{7}}{\sqrt{3}};\ \ \sqrt{3}, 0;$

   $-\sqrt{3}, 0;\ \pm\sqrt{7}, 0; \dfrac{3}{\sqrt{7}};$

   $x = \pm\dfrac{3}{\sqrt{7}};\ y = \pm\dfrac{2}{\sqrt{3}}x.$

5. $Tan^{-1}\frac{4}{3},\ Tan^{-1} -\frac{4}{3};$

   $\dfrac{x}{\sqrt{3}} \pm \dfrac{y}{\sqrt{2}} = 0.$

6. $5x^2 - 4y^2 = 125.$

8. $3x^2 - y^2 = 3a^2.$

9. $153x^2 - 425y^2 = 450.$

10. $9x^2 - 16y^2 = 20.$

20. $\dfrac{y^2}{4} - \dfrac{x^2}{3} = 1;\ \dfrac{y^2}{18} - \dfrac{x^2}{16} = 1;$

    $\dfrac{y^2}{36} - \dfrac{x^2}{108} = 1;\ \dfrac{y^2}{25} - \dfrac{x^2}{11} = 1$

EXERCISE 167. (Page 185).

1. $y^2 = 2x;\ \dfrac{x^2}{1} + \dfrac{9y^2}{5} = 1;$

   $\dfrac{x^2}{36} - \dfrac{y^2}{45} = 1.$

2. $x^2 = -16y.$

3. $\dfrac{x^2}{7} + \dfrac{y^2}{16} = 1.$

4. $\frac{-x^2}{4} + \frac{7y^2}{36} = 1.$

5. $11x^2 + 96xy + 39y^2 - 138x$
$-284y + 251 = 0.$

6. Circle, ellipse, hyperbola.

7. $y + 2x + 1 = 0; \quad (\frac{1}{2}, -2);$
$2y = x + 8; \; 8, 8.$

9. $y^2 - 8x - 8y = 0.$

11. $\pm \frac{3}{26}\sqrt{65}, \mp \frac{2}{39}\sqrt{195}.$

12. $y = \pm (x + 2a).$

13. $x^2 - 2xy + y^2 - 6x - 14y +$
$39 = 0.$

14. $43x^2 + 12xy + 48y^2 + 184x$
$- 192y + 400 = 0.$

15. $76x^2 - 72xy - 59y^2 - 136x$
$- 272y - 340 = 0.$

16. $-2, 0; \; 5; \; -2, -\frac{3}{2}; \; \frac{8}{3}.$

17. $1, 0; \; -1, 0; \; 3; \; \frac{8}{3}\sqrt{2}, 0;$
$-\frac{8}{3}\sqrt{2}, 0; \; \frac{8}{9}.$

18. $x = \pm \frac{2}{\sqrt{5}}; \; \frac{x}{2} \pm \frac{y}{4} = 0.$
$x = \pm \frac{2}{3\sqrt{5}}; \; x \pm 2y = 0.$
$y = \pm \sqrt{5}; \; x \pm 2y = 0.$

19. $3x - 4y = 25; \; 4x + 3y = 0.$

20. $x - y + 2 = 0; \; x + y - 6 = 0.$

21. $9x + 8y = 25; \; 8x - 9y + 10 = 0.$

22. $5x + 3y + 16 = 0; \; 3x - 5y$
$+ 30 = 0.$

23. $9x^2 - 7y^2 = 63.$

24. $y - x\sqrt{3} = \pm 12; \; 3x - y\sqrt{3}$
$+ 4 = 0. \quad y = x\sqrt{3} \pm 2\sqrt{5};$
$y = x\sqrt{3} \pm \sqrt{46}.$

26. $3x + 2y = 16; \; x - y + 6 = 0;$
$6x + y = 12; \; 6x - y = 16.$

27. $27, 36; \; -\frac{4}{3}, -2; \; 2, 2; \; \frac{9}{20}, -3.$

28. $\pm \tan^{-1} \frac{1}{2}\sqrt{5}.$

31. $4x + 7y = 25; \; 3x - 7y + 6 = 0;$
$6x + 24y = 25; \; x - y = 20.$

33. $3x + 4y \pm 25 = 0.$

34. $4x - 3y + \frac{9}{2} = 0.$

35. $6x + 12y \pm 25 = 0.$

36. $y = \frac{-3x}{2} \pm \frac{1}{12}\sqrt{1625}.$

37. $x\sqrt{17} - y\sqrt{17} \pm 1 = 0.$

38. $bx + ay = \pm ab\sqrt{2}.$

39. $x - y = 2.$

40. $\sqrt{\frac{b}{a}}.$

41. $x + 2y = 4.$

44. An hyperbola.

DATE DUE SLIP

RECOMMENDED FOR USE
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F255

o

$\tan 45^\circ = 1$

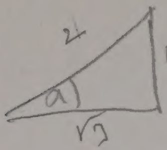
$45 = \tan^{-1} 1$

$$\begin{array}{r} 1.73 \\ \sqrt{3.0000} \\ 1 \\ \hline 200 \\ 189 \\ \hline 1100 \end{array}$$

27

343

$$1.73 \overline{)1.0000}\ .5$$



$\tan \alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = \tan^{-1} \frac{1}{\sqrt{3}}$

$=$

Required for trig x anal. geom

Brief treatment
- st. line
- circle
- parabola
- ellipse } particularly brief
- hyperbola }

omit:
- oblique coordinates
- normal form of st line (§ 35)
- perp. dist. from pt. to line (45-53
- rotation of axes, pole, polars, trans. f
- external pt. (except perhaps § 102)
- §§ 88, 89, 90, 97, 103, 107-110.

Proofs shd not be memorized.

One form of linear functions,
slope-intercept

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